101—102

工程數學歷屆試題

機械所 土木所

勘誤表

1. 102 台灣大學機械工程研究所

1. (15%) Let C be the unit circle $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$.
   (a) (7%) For any real constant r , determine $\int_C \frac{e^{rz}}{z}\,dz$.
   (b) (8%) Apply $z = \cos\theta + i\sin\theta$ and the result of (a), determine

$$\int_0^\pi e^{r\cos\theta}\cos(r\sin\theta)\,d\theta$$

《喻超凡, 喻超弘 102台大機械》

《解》☞

(a) 令 $f(z) = \frac{e^{rz}}{z}$, 故 $f(z)$ 在 C 內具有 $z = 0$ 的一階 pole , 則

$$\text{Res}f(0) = e^{rz}\Big|_{z=0} = 1$$

故

$$\int_C \frac{e^{rz}}{z}\,dz = 2\pi i\,\text{Res}f(0) = 2\pi i$$

(b) $C : z = e^{i\theta}$, 故 $dz = e^{i\theta}\,i\,d\theta$,

$$\begin{aligned}
\oint_C \frac{e^{rz}}{z}\,dz &= \int_{-\pi}^{\pi} \frac{e^{re^{i\theta}}}{e^{i\theta}}\,e^{i\theta}\,i\,d\theta \\
&= \int_{-\pi}^{\pi} e^{r\cos\theta}e^{ir\sin\theta}\,i\,d\theta \\
&= \int_{-\pi}^{\pi} e^{r\cos\theta}\{\cos(r\sin\theta) + i\sin(r\sin\theta)\}i\,d\theta \\
&= 2\int_0^{\pi} e^{r\cos\theta}\cos(r\sin\theta)\,i\,d\theta \\
&= 2\pi i
\end{aligned}$$

故

$$\int_0^\pi e^{r\cos\theta}\cos(r\sin\theta)\,i\,d\theta = \pi$$

其中 $\mathscr{L}\{w(t)\} = W(s)$, 整理可得

$$-2sW(s) - s^2\frac{dW(s)}{ds} + (1-2n)sW(s) - \frac{d}{ds}W(s) = 0$$

即

$$\frac{dW(s)}{ds} + (1+2n)\frac{s}{s^2+1}W(s) = 0$$

故

$$\frac{dW(s)}{W(s)} = -(1+2n)\frac{s}{s^2+1}ds$$

兩端積分可得

$$\ln|W(s)| = -\frac{1}{2}(1+2n)\ln|s^2+1| + c_1$$

即

$$W(s) = \frac{c}{(s^2+1)^{\frac{1+2n}{2}}}$$

(c) 因

$$\begin{aligned} W(s) &= \frac{c}{(s^2+1)^{\frac{1+2n}{2}}} \\ &= c\{\frac{1}{s^{1+2n}}(1+\frac{1}{s^2})^{-(\frac{1}{2}+n)}\} \\ &= c\{\frac{1}{s^{1+2n}}\sum_{m=0}^{\infty}\binom{-(\frac{1}{2}+n)}{m}(\frac{1}{s^2})^m\} \\ &= c\sum_{m=0}^{\infty}\binom{-(\frac{1}{2}+n)}{m}\frac{1}{s^{1+2n+2m}} \end{aligned}$$

故

$$\begin{aligned} w(t) &= \mathscr{L}^{-1}\{W(s)\} \\ &= \mathscr{L}^{-1}\{c\sum_{m=0}^{\infty}\binom{-(\frac{1}{2}+n)}{m}\frac{1}{s^{1+2n+2m}}\} \\ &= c\sum_{m=0}^{\infty}\binom{-(\frac{1}{2}+n)}{m}\frac{t^{2m+2n}}{(2m+2n)!} \end{aligned}$$

因此

$$u(x\,,\,y) = (u_1 - u_0)y + u_0 + \sum_{n=1}^{\infty} \frac{2(-u_0 + u_1 \cos n\pi)}{n\pi} e^{-n\pi x} \sin n\pi y$$

7. 已知週期函數 $f(t)$ 在某一週期內的定義爲

$$f(t) = 1 + t\ ;\quad -1 < t < 1$$

(1) 試求週期函數 $f(t)$ 的傅立葉級數 (Fourier series) 。(8%)

(2) 利用 $f(t)$ 的傅立葉級數求出下列級數之和：

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(3) 假設 F 代表由 $f(t)$ 之傅立葉級數定義於區間 $(-\infty\,,\,\infty)$ 的函數，試求 $F(1) + F(-5) - 3F(0) =$? (3%) 《喻超凡, 喻超弘 102 台大生機》

《解》 ☞

(1) 因

$$t = \sum_{n=1}^{\infty} b_n \sin(n\pi t)$$

其中

$$b_n = 2\int_0^1 t \sin(n\pi t)\, dt = -\frac{2\cos n\pi}{n\pi} = \frac{(-1)^{n+1}2}{n\pi}$$

故

$$f(t) = 1 + t = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2}{n\pi} \sin(n\pi t) \tag{1}$$

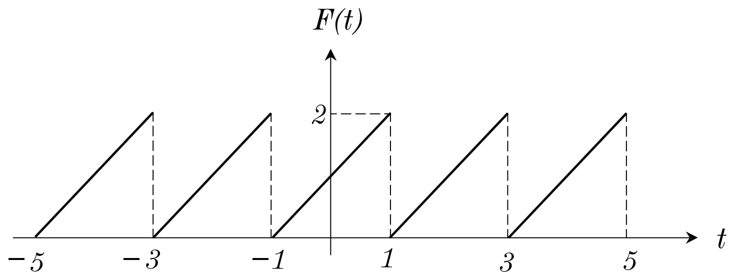
(2) 令 $t = \frac{1}{2}$ 代回 (1) 式中

$$1 + \frac{1}{2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2}{n\pi} \sin\frac{n\pi}{2}$$

即

$$1 - \frac{1}{3} + \frac{1}{5} + \cdots = \frac{\pi}{4}$$

(3)



$$F(1) + F(-5) - 3F(0) = 1 + 1 - 3 \times 1 = -1$$

則曲線 C 爲 Σ 之邊界曲線，同時亦爲 S 之邊界曲線 (如圖)。又因爲 Σ 的單位法向量 $\vec{n}$ 爲朝外的方向，故 S_1 的單位法向量 $\vec{n} = -\vec{k}$，且

$$(\nabla \times \vec{F}) \cdot \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & xy & -xyz \end{vmatrix} \cdot (-\vec{k}) = -(y+1)$$

故由 Stokes's 定理可得

$$\begin{aligned} \iint_\Sigma (\nabla \times \vec{F}) \cdot d\vec{A} &= \iint_\Sigma (\nabla \times \vec{F}) \cdot \vec{n} dA = \oint_C \vec{F} \cdot d\vec{r} \\ &= \iint_S (\nabla \times \vec{F}) \cdot \vec{n} dA = \iint_S -(y+1) dA \\ &= -(\overline{y}+1)A = -(0+1) \times \pi \times 3^2 = -9\pi \end{aligned}$$

5. (30%) Solve the following partial differential equation :

$$u_{xx} + u_{yy} = 0\ ,\ 0 < x < \pi\ ,\ 0 < y < 1$$

$u(x\,,\,0) = \sin x\,\cos x$, $0 \le x \le \pi$, $u(x\,,\,1) = 1$, $0 \le x \le \pi$,
$u(0\,,\,y) = y$, $0 \le y \le 1$, $u(\pi\,,\,y) = y$, $0 \le y \le 1$

(a) What is the type of the second-order partial differential equation given above? (Hyperbolic, parabolic, or elliptic) (2%)

(b) What is the physical nature of the partial differential equation given above ? (wave propagation, equilibrium or heat/diffusion) (2%)

(c) What is the type of boundary condition given above ? (Dirichlet, Neumann, or Robin) (20%)

(d) Is the partial differential equation (PDE) given above a linear PDE ? (2%)

(e) Is the partial differential equation (PDE) given above homogeneous ? (2%)

(f) Obtain the explicit solution of the problem given above. (15%)

(g) What is the maximum value of $u(x\,,\,y)$? (5%)

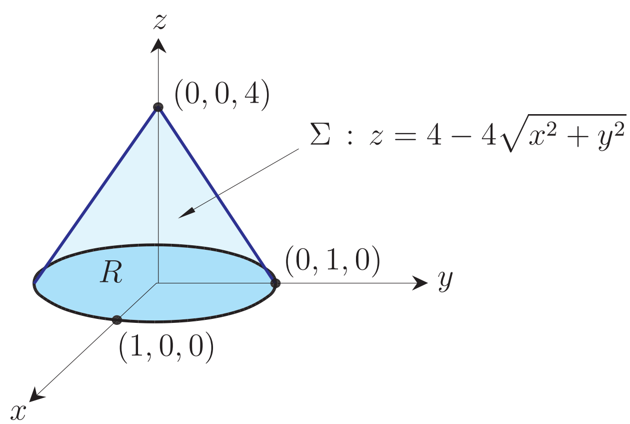
《喻超凡, 喻超弘 102台科大機械》

《解》☞

(a) 因 $b^2 - 4ac < 0$，故爲 elliptic type。

5. 一曲面 Σ 由函數方程式 $\Sigma : z = 4 - 4\sqrt{x^2+y^2}$ 且 $x \geq 0$；$y \geq 0$；$4 \geq z \geq 0$ 所定義, 請計算
(1) 曲面 Σ 之面積 A。(10%)
(2) 曲面 Σ 與 $x = 0$；$y = 0$；$z = 0$ 三平面所夾的體積。(10%)

《喻超凡, 喻超弘 102台科大營建》



《解》 ☞

(1) Σ 投影到 x-y 平面的區域為 $R : x^2 + y^2 = 1$ 且 $x \geq 0$、$y \geq 0$，令

$$\phi = x^2 + y^2 - (1 - \frac{z}{4})^2 = 0$$

故單位法向量為

$$\vec{n} = \frac{x\vec{i} + y\vec{j} + \frac{1}{4}(1-\frac{z}{4})\vec{k}}{\sqrt{x^2+y^2+\frac{1}{16}(1-\frac{z}{4})^2}}$$

故

$$dA = \frac{dxdy}{|\vec{k}\cdot\vec{n}|} = \frac{\sqrt{x^2+y^2+\frac{1}{16}(1-\frac{z}{4})^2}}{\frac{1}{4}(1-\frac{z}{4})}dxdy = \sqrt{17}\,dxdy$$

因此

$$A = \iint_{\Sigma} dA = \iint_{R} \sqrt{17}\,dxdy = \sqrt{17} \times \frac{1}{4} \times \pi$$

第2步調整 x 的次冪

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}-\sum_{n=1}^{\infty}(n-1+r)a_{n-1}x^{n+r-1}$$
$$-\sum_{n=0}^{\infty}2(n+r)a_n x^{n+r-1}+\sum_{n=1}^{\infty}2a_{n-1}x^{n+r-1}=0$$

第3步合併同次項

$$\{r(r-1)-2r\}a_0x^{r-1}+\sum_{n=1}^{\infty}\{(n+r)(n+r-3)a_n-(n+r-3)a_{n-1}\}x^{n+r-1}=0$$

比較係數可得

$$r(r-3)a_0=0 \tag{1}$$
$$(n+r)(n+r-3)a_n-(n+r-3)a_{n-1}=0 \quad ;\ (n=1\,,\,2\,,\,3\cdots) \tag{2}$$

因 (1) 式中 $a_0\neq 0$, 故可得指標方程式爲

$$r(r-3)=0 \tag{3}$$

解 (3) 式可得指標根爲 $r_1=0$ 、$r_2=3$, 且 $r_2-r_1=3\in\mathbb{Z}$ (整數) , 將 $r=r_0=0$ 代回 (2) 式可得

$$n(n-3)\,a_n-(n-3)a_{n-1}=0 \quad ;\ (n=1\,,\,2\,,\,3\cdots)$$

故 $n=1$ 時 , $a_1=a_0$, $n=2$ 時

$$a_2=\frac{1}{2}a_1=\frac{1}{2}a_0$$

$n=3$ 自動滿足 , 且

$$a_n=\frac{a_{n-1}}{n} \quad ;\ (n=4\,,\,5\,,\,6\cdots) \tag{4}$$

由 (4) 式可得

$$\begin{aligned} n=4 &\quad\Rightarrow\quad a_4=\frac{a_3}{4} \\ n=5 &\quad\Rightarrow\quad a_5=\frac{a_4}{5}=\frac{3!a_3}{5!} \\ n=6 &\quad\Rightarrow\quad a_6=\frac{a_5}{6}=\frac{3!a_3}{6!} \\ \cdots &\quad \cdots \quad \cdots \end{aligned}$$

10. 102 交通大學機械工程研究所 (甲)

1. (8%) Let

$$f(x) = 1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots + \frac{x^{96}}{96!} - \frac{x^{97}}{97!} - \frac{x^{98}}{98!} + \frac{x^{99}}{99!}$$

Find the closest value of maximum $f(x)$ for $x \in \mathbb{R}$ and $-3 < x < 3$.

(a) 0 (b) 1 (c) 2 (d) e (e) $\pi/2$ 《喻超凡, 喻超弘 102交大機械甲》

《解》 ☞ 因

$$\begin{aligned} f(x) &= 1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots + \frac{x^{96}}{96!} - \frac{x^{97}}{97!} - \frac{x^{98}}{98!} + \frac{x^{99}}{99!} \\ &= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots - \frac{x^{98}}{98!}) - (x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots - \frac{x^{99}}{99!}) \\ &\approx \cos x - \sin x \\ &= \sqrt{2}(\cos\frac{\pi}{4}\cos x - \sin x \sin\frac{\pi}{4}) \\ &= \sqrt{2}\cos(x + \frac{\pi}{4}) \end{aligned}$$

當 $x = -\frac{\pi}{4}$ 時, 具有極大值為 $f(\frac{\pi}{4}) \approx \sqrt{2}$, 故選 (e)

2. (8%) $(\sqrt{2i} - 1)^{2013} = ?$

(a) 0 (b) 1 (c) i (d) π (e) $2013 - 2013i$

《喻超凡, 喻超弘 102交大機械甲》

《解》 ☞ 因 $(1+i)^2 = 1 + 2i - 1 = 2i$, 故

$$(\sqrt{2i} - 1)^{2013} = (1 + i - 1)^{2013} = i^{2012} \times i = i$$

7. Please solve the following differential equations :

(a) $\dfrac{dy}{dx} = \dfrac{1}{x+y^2}$, subject to $y(-2) = 0$ (7%)

(b) $y'' = 2x(y')^2$ (7%)

(c) Given that $y_1 = x^3$ is a solution of $x^2y'' - 6y = 0$. Use reduction of order to find a second solution on the interval $0 < x < \infty$. (9%)

《喻超凡, 喻超弘 102交大機械甲》

《解》☞

(a) 原式可改寫成

$$\frac{dx}{dy} = x + y^2 \ \Rightarrow\ \frac{dx}{dy} - x = y^2$$

積分因子爲 $I = e^{-y}$, 故

$$Ix = \int y^2 e^{-y}\, dy = e^{-y}(-y^2 - 2y - 2) + c$$

因此

$$x = -y^2 - 2y - 2 + ce^y$$

再由 $y(-2) = 0$, 可得 $c = 0$, 故

$$x = -y^2 - 2y - 2$$

(b) 令 $p = y'$, 代回 ODE 中可得

$$p' = 2xp^2 \ \Rightarrow\ \frac{dp}{p^2} = 2x\, dx$$

兩端積分可得

$$-\frac{1}{p} = x^2 + c_1 \ \Rightarrow\ p = y' = -\frac{1}{x^2 + c_1}$$

故

$$y = -\int \frac{1}{x^2 + c_1} dx = -\frac{1}{\sqrt{c_1}} \tan^{-1} \frac{x}{\sqrt{c_1}} + c_2$$

令 $c_1^* = \sqrt{c_1}$, 整理可得

$$y = -\frac{1}{c_1^*} \tan^{-1} \frac{x}{c_1*} + c_2$$

令

$$S=\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{30}}\\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{30}}\\ 0 & -\frac{1}{\sqrt{6}} & \frac{5}{\sqrt{30}}\end{bmatrix}$$

故 S 為正交矩陣，且

$$D=S^{-1}AS=\begin{bmatrix}-3 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 7\end{bmatrix}$$

4. Solve the problem using Laplace transform. (15%)

$$y''+4y'+4y=1+\delta(t-1)\ ,\ y(0)=0\ ,\ y'(0)=2$$

《喻超凡, 喻超弘 102交大機械乙》

《解》 ☞ 對 ODE 的取 L –T 可得

$$s^2Y(s)-sy(0)-y'(0)+4\{sY(s)-y(0)\}+4Y(s)=\frac{1}{s}+e^{-s}$$

其中 $\mathscr{L}\{y(t)\}=Y(s)$，整理可得

$$(s^2+4s+4)Y(s)=2+\frac{1}{s}+e^{-s}$$

即

$$Y(s)=\frac{2s+1}{s(s+2)^2}+\frac{1}{(s+2)^2}e^{-s}=\frac{1}{4s}+\frac{3}{2(s+2)^2}-\frac{1}{4(s+2)}+\frac{1}{(s+2)^2}e^{-s}$$

故

$$y(t)=\mathscr{L}^{-1}\{Y(s)\}=\frac{1}{4}+\frac{3}{2}te^{-2t}-\frac{1}{4}e^{-2t}+(t-1)e^{-2(t-1)}H(t-1)$$

故
$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = e^{2t} + 2e^{-t} + \{e^{2(t-\pi)} - e^{-(t-\pi)}\}H(t-\pi)$$

3. (20%) If the distance between the plane $Ax - 2y + z = d$ and the plane containing lines
$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$
and
$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$$
is $\sqrt{6}$, then find A and d. 《喻超凡, 喻超弘 102交大機械丙》

《解》☞ $Ax - 2y + z = d$ 的法向量為
$$\vec{n}_1 = (A\,,\,-2\,,\,1)$$
另外一平面法向量為
$$\vec{n}_2 = (2\,,\,3\,,\,4) \times (3\,,\,4\,,\,5) = (-1\,,\,2\,,\,-1)$$
故 $\vec{n}_1 \parallel \vec{n}_2$, 即
$$(A\,,\,-2\,,\,1) = k(-1\,,\,2\,,\,-1)$$
則 $A = 1$, 又直線上的一點 $P(1\,,\,2\,,\,3)$ 到平面的距離為
$$\left|\frac{1 - 2\times 2 + 3 - d}{\sqrt{1^2 + (-2)^2 + 1^2}}\right| = \sqrt{6}$$
故可得 $d = \pm 6$ 。

4. (20%) If $\vec{A} = x\vec{i} + y\vec{j} + (-2z + \frac{1}{2}z^2)\vec{k}$, evaluate $\int_S (\vec{A}\cdot\vec{n})\,dS$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy plane, $\vec{n}$ is the unit outward normal vector of the surface of the sphere. 《喻超凡, 喻超弘 102交大機械丙》

《解》☞

$$I=\int_C 3x^2dx+2yzdy+y^2dz=\int_{(0,1,2)}^{(1,-1,3)} d(x^3+y^2z)=(x^3+y^2z)\Big|_{(0,1,2)}^{(1,-1,3)}=2$$

3. (17%) Find the Fourier transform of $f(x)=1$ if $|x|=2$ and $f(x)=0$ otherwise.
《喻超凡, 喻超弘 102交大機械丁》

《解》☞ 題目應改成

$$f(x)=\begin{cases}1 & ;\ |x|\le 2\\ 0 & ;\ \text{otherwise}\end{cases}$$

故

$$\begin{aligned}\mathscr{F}\{f(x)\} &= \int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx\\ &= \int_{-2}^{2}(\cos wx - i\sin wx)\,dx\\ &= 2\int_0^2 \cos wx\,dx = 2\frac{\sin wx}{w}\Big|_0^2\\ &= \frac{2\sin 2w}{w}\end{aligned}$$

4. A damped mass-spring system is actuated by an external force. The model of system is described by differential equation :

$$y''+5y'+6y=12u(t-1)-6u(t-3)\ ,\ y(0)=0\ ,\ y'(0)=0$$

(1) (7%) Determine the response of the system.
(2) (7%) State the steady-state of the response on $t>3$
《喻超凡, 喻超弘 102交大機械丁》

《解》☞

《解》☞ 原式可改寫成

$$\{xD^2 + (x^2-3)D - 2x\}y = 0$$

可因式分解成

$$(xD-3)(D+x)y = 0$$

令 $z = (D+x)y$, 故可得

$$(xD-3)z = 0 \;\Rightarrow\; xz' - 3z = 0$$

或

$$\frac{dz}{z} - \frac{3}{x}dx = 0$$

兩端積分可得

$$\ln|z| - 3\ln|x| = c_1^*$$

即 $z = c_1x^3$, 故

$$(D+x)y = z = c_1x^3$$

爲一階線性 , 積分因子爲 $I = e^{\int x\,dx} = e^{\frac{x^2}{2}}$, 故

$$Iy = \int c_1x^3\,e^{\frac{x^2}{2}}\,dx = c_1\,e^{\frac{x^2}{2}}(x^2-2) + c_2$$

即

$$y(x) = c_1(x^2-2) + c_2e^{\frac{-x^2}{2}}$$

3. $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

(a) Find A^{-1} using the method of Gauss-Jordan elimination. (8%)

(b) Find the eigenvalues of A^{-1}. (8%)

(c) Find the eigenvalues of A^5. (7%)

(d) If $D = P^{-1}AP$ is a diagonal matrix, find the matrix P. (7%)

《喻超凡, 喻超弘 102交大土木甲》

《解》☞

(a) $$\left[\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_{13}} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 0 & 1 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 \end{array}\right]$$

故煤層的平面與水平面的夾角爲

$$\cos\theta = \vec{n}\cdot\vec{k} = \frac{24}{\sqrt{649}} \;\Rightarrow\; \theta = \cos^{-1}\{\frac{24}{\sqrt{649}}\}$$

3. 試計算向量 $\vec{V} = xz^2\vec{i} + x^2y\vec{j} + \frac{1}{3}x^3\vec{k}$ 的 divergence 及 curl。(10%)

《喻超凡, 喻超弘 102交大土木丁》

《解》☞

$$\nabla\cdot\vec{V} = z^2 + x^2$$

$$\nabla\times\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^2 & x^2y & \frac{1}{3}x^3 \end{vmatrix} = (-x^2 + 2xz)\vec{j} + 2xy\vec{k}$$

4. (15%) Which one(s) of the following two ODEs are linear, which one(s) are non-linear ?

(a) $(2x^3 + x)y' + e^x y = 0$

(b) $(y'')^2 + xy' = 0$

(c) Let y_1 and y_2 be two solutions for (a), is $c_1y_1 + c_2y_2$, where c_1 and c_2 are constants, also a solution of the differential equation (a) ? Explain why ?

《喻超凡, 喻超弘 102交大土木丁》

《解》☞ (a) 爲線性 ODE , (b) 爲非線性 ODE , 若 y_1 、y_2 爲 (a) ODE 的解, 則

$$\begin{cases} (2x^3 + x)y_1' + e^x y_1 = 0 \\ (2x^3 + x)y_2' + e^x y_2 = 0 \end{cases}$$

故

$$(2x^3 + x)\{c_1y_1 + c_2y_2\}' + e^x\{c_1y_1 + c_2y_2\} = 0$$

即

$$c_1\{(2x^3 + x)y_1' + e^x y_1\} + c_2\{(2x^3 + x)y_2' + e^x y_2\} = 0$$

故

$$a_{2n} = (-1)^n \frac{4^n a_0}{(2n)!} \ , \ a_{2n+1} = 0$$

因此

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+r} \Big|_{r=r_1=-1} \\ &= \sum_{n=0}^{\infty} a_{2n} x^{2n-1} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n} \\ &= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{2n-1} \\ &= a_0 \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2x)^{2n} = a_0 \frac{\cos 2x}{x} \end{aligned}$$

(c) 故 ODE 的通解爲

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1^* \frac{\sin 2x}{2x} + c_2^* \frac{\cos 2x}{x}$$

8. (10%) Solve the given initial value problem :

$$y'' + 9y = \begin{cases} 6\sin t & ; \text{if } 0 \le t \le \pi \\ 0 & ; \text{if } t > \pi \end{cases} \quad ; \quad y(0) = y'(0) = 0$$

《**喻超凡, 喻超弘** 102交大土木丁》

《**解**》☞ 令

$$f(t) = \begin{cases} 6\sin t & ; \text{if } 0 \le t \le \pi \\ 0 & \text{if } t > \pi \end{cases} = 6\sin t\,\{H(t) - H(t-\pi)\}$$

故

$$\mathscr{L}\{f(t)\} = 6\frac{1}{s^2+1} - e^{-\pi s}\mathscr{L}\{6\sin(t+\pi)\} = \frac{6}{s^2+1} + \frac{6}{s^2+1}e^{-\pi s}$$

對 ODE 兩端取 L–T 可得

$$s^2 Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{6}{s^2+1} + \frac{6}{s^2+1}e^{-\pi s}$$

其中 $\mathscr{L}\{y(t)\} = Y(s)$, 整理可得

$$\begin{aligned} Y(s) &= \frac{6}{(s^2+1)(s^2+9)} + \frac{6}{(s^2+1)(s^2+9)}e^{-\pi s} \\ &= \frac{3}{4(s^2+1)} - \frac{3}{4(s^2+9)} + \{\frac{3}{4(s^2+1)} - \frac{3}{4(s^2+9)}\}e^{-\pi s} \end{aligned}$$

(b) 由

$$u(x\,,\,0) = x = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

故

$$A_n = 2\int_0^1 x\,\sin(n\pi x)\,dx = \boxed{\frac{-2\cos n\pi}{n\pi}}$$

則

$$u(x\,,\,t) = \sum_{n=1}^{\infty} \boxed{\frac{-2\cos n\pi}{n\pi}} \sin(n\pi x)$$

(c) 令 $u(x\,,\,t) = \phi(x\,,\,t) + e^{-t}x$, 代回 PDE 中可得

$$\frac{\partial \phi}{\partial t} - e^{-t}x = \frac{\partial^2 \phi}{\partial x^2}$$

故

$$u(0\,,\,t) = \phi(0\,,\,t) = 0$$

$$u(1\,,\,t) = \phi(1\,,\,t) + e^{-t} = e^{-t} \;\Rightarrow\; \phi(1\,,\,t) = 0$$

$$u(x\,,\,0) = \phi(x\,,\,0) + x = x \;\Rightarrow\; \phi(x\,,\,0) = 0$$

因 $\phi(0\,,\,t) = \phi(1\,,\,t) = 0$, 故令

$$\phi(x\,,\,t) = \sum_{n=1}^{\infty} a_n(t)\sin(n\pi x)$$

$$x = \sum_{n=1}^{\infty} q_n \sin(n\pi x)$$

其中

$$q_n = 2\int_0^1 x\,\sin(n\pi x)\,dx = -\frac{\cos n\pi}{n\pi}$$

代回 PDE 中可得

$$\sum_{n=1}^{\infty} a_n'(t)\sin(n\pi x) - e^{-t}\sum_{n=1}^{\infty} q_n \sin(n\pi x) = \sum_{n=1}^{\infty} a_n(t)(-n^2\pi^2)\sin(n\pi x)$$

即

$$\sum_{n=1}^{\infty}\{a_n'(t) + (n\pi)^2 - e^{-t}q_n\}\sin(n\pi x) = 0$$

故

$$a_n'(t) + (n\pi)^2 a_n(t) = e^{-t}q_n$$

可解得 $a_n(t) = A_n e^{-(n\pi)^2 t} + \dfrac{q_n}{(n\pi)^2 - 1}e^{-t}$, 則

$$\phi(x\,,\,t) = \sum_{n=1}^{\infty}\{A_n e^{-(n\pi)^2 t} + \frac{q_n}{(n\pi)^2-1}e^{-t}\}\sin(n\pi x) \tag{1}$$

3. (20%) Consider a vibrating elastic bar in figure 2. The boundary at $x = 0$ and $x = L$ are called free-end conditions. Solve the wave equation.

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \text{ , } 0 < x < L \text{ , } t > 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \text{ , } \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0 \text{ , } u(x\,,\,0) = x \text{ , } \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

《喻超凡, 喻超弘 102中興機械》

《解》☞ 因 $\frac{\partial u}{\partial x}(0\,,\,t) = 0$ 、$\frac{\partial u}{\partial x}(L\,,\,t) = 0$, 故特徵函數為 $\left\{1\ ,\ \cos\frac{n\pi x}{L}\right\}_{n=1}^{\infty}$, 則令

$$u(x\,,\,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos\frac{n\pi x}{L}$$

代回 PDE 中可得

$$\alpha^2 \sum_{n=1}^{\infty} a_n(t)\left(-\frac{n^2\pi^2}{L^2}\right)\cos\frac{n\pi x}{L} = a_0''(t) + \sum_{n=1}^{\infty} a_n''(t)\cos\frac{n\pi x}{L}$$

即

$$a_0''(t) + \sum_{n=1}^{\infty}\left\{a_n''(t) + \left(\frac{\alpha n\pi}{L}\right)^2 a_n(t)\right\}\cos\frac{n\pi x}{L} = 0$$

即

$$\begin{cases} a_0''(t) = 0 \\ a_n''(t) + \left(\frac{\alpha n\pi}{L}\right)^2 a_n(t) = 0 \end{cases} \Rightarrow \begin{cases} a_0(t) = A_0 + B_0 t \\ a_n(t) = A_n\cos\frac{\alpha n\pi t}{L} + B_n\sin\frac{\alpha n\pi t}{L} \end{cases}$$

故

$$u(x\,,\,t) = A_0 + B_0 t + \sum_{n=1}^{\infty}\left\{A_n\cos\frac{\alpha n\pi t}{L} + B_n\sin\frac{\alpha n\pi t}{L}\right\}\cos\frac{n\pi x}{L} \qquad (1)$$

且

$$u(x\,,\,0) = x = A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}$$

故

$$A_0 = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$$

$$A_n = \frac{2}{L}\int_0^L x\,\cos\frac{n\pi x}{L}\,dx = \frac{2L(-1+\cos n\pi)}{n^2\pi^2}$$

且

$$\frac{\partial u}{\partial t}(x\,,\,0)=0=B_0+\sum_{n=1}^{\infty}B_n\frac{\alpha n\pi}{L}\cos\frac{n\pi x}{L}$$

故 $B_0=0$、$B_n=0$, 將 A_0、A_n、B_0、B_n 代回 (1) 式, 即為解。

4. (20%) Solve the system of differential equations

$$\begin{cases}\dfrac{d^2x}{dt^2}+\dfrac{d^2y}{dt^2}=t^2\\ \dfrac{d^2x}{dt^2}-\dfrac{d^2y}{dt^2}=4t\end{cases}$$

$x(0)=0,\ x'(0)=0\ ,\ y(0)=0\ ,\ y'(0)=0$

《**喻超凡, 喻超弘** 102中興機械》

《解》 ☞ 因

$$\begin{cases}\dfrac{d^2x}{dt^2}+\dfrac{d^2y}{dt^2}=t^2\\ \dfrac{d^2x}{dt^2}-\dfrac{d^2y}{dt^2}=4t\end{cases}$$

兩式相加, 可得

$$2\frac{d^2x}{dt^2}=t^2+4t$$

兩積分2次可得

$$x(t)=\frac{t^3}{3}+\frac{t^4}{24}+c_1t+c_2$$

同理兩式相減，可得

$$2\frac{d^2y}{dt^2}=t^2-4t$$

兩積分2次可得

$$y(t)=-\frac{t^3}{3}+\frac{t^4}{24}+d_1t+d_2$$

由 $x(0)=0=c_2,\ x'(0)=0=c_1\ ,\ y(0)=0=d_2\ y'(0)=0=d_1$，故

$$x(t)=\frac{t^3}{3}+\frac{t^4}{24}$$

$$y(t)=-\frac{t^3}{3}+\frac{t^4}{24}$$

20. 102 中興大學精密工程研究所

1. Use the idea of matrix to solve the following system of linear equations. (15%)

$$\begin{cases} x_1+2x_2+x_3+x_4+x_5=0 \\ -x_1+x_3+x_4+2x_5=0 \\ x_2+3x_3+4x_5=0 \\ -3x_1+x_2+4x_5=0 \end{cases}$$

《**喻超凡, 喻超弘** 102中興精密》

《**解**》☞ $\begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 3 & 0 & 4 \\ -3 & 1 & 0 & 0 & 4 \end{bmatrix} \xrightarrow{R_{12}^{(1)}R_{14}^{(-3)}} \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 3 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 7 & 3 & 3 & 7 \end{bmatrix} \xrightarrow{R_{23}} \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 2 & 2 & 2 & 3 \\ 0 & 7 & 3 & 3 & 7 \end{bmatrix}$

$$\xrightarrow{R_{23}^{(-2)}R_{24}^{(-7)}} \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & -4 & 2 & -5 \\ 0 & 0 & -18 & 3 & -21 \end{bmatrix} \xrightarrow{R_{34}^{(-9/2)}} \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & -4 & 2 & -5 \\ 0 & 0 & 0 & -6 & 3/2 \end{bmatrix}$$

令 $x_5=8c$, 則 $x_4=2c$ 、$x_3=-9c$ 、$x_2=-5c$ 、$x_1=9c$, c 為任意數 。

2. The circuit has the output voltage $q(t)/C$ as the following equation. The current and charge on the capacitor are zero at time zero ($q(0)=q'(0)=0$). Please determine the output voltage response to transient modeled by $\delta(t)$. (15%)

$$q''+20q'+25q=\delta(t)$$

《**喻超凡, 喻超弘** 102中興精密》

《**解**》☞ 對 ODE 兩端取 L–T , 可得

$$s^2Q(s)-sq(0)-q'(0)+20\{sQ(s)-q(0)\}+25Q(s)=1$$

其中 $\mathscr{L}\{q(t)\} = Q(s)$, 整理可得

$$(s^2 + 20s + 25)Q(s) = 1$$

即

$$Q(s) = \frac{1}{s^2 + 20s + 25} = \frac{1}{(s+10)^2 - 75}$$

故

$$q(t) = \mathscr{L}^{-1}\{Q(s)\} = e^{-10t}\frac{1}{\sqrt{75}}\sinh(\sqrt{75}t)$$

3. The point $A(1\,,\,-2\,,\,1)$, $B(0\,,\,1\,,\,6)$, and $C(-3\,,\,4\,,\,-2)$ from the vertices of a triangle. Please calculate the cosine of the angle between AB and BC. (15%)

《喻超凡, 喻超弘 102中興精密》

《解》 ☞ $\overrightarrow{AB} = (-1\,,\,3\,,\,5)$ 、$\overrightarrow{BC} = (-3\,,\,3\,,\,-8)$, 故

$$\cos\theta = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}\cdot\frac{\overrightarrow{BC}}{|\overrightarrow{BC}|} = \frac{3+9-40}{\sqrt{35}\,\sqrt{82}} = -\frac{28}{\sqrt{2870}}$$

4. Please find the mass and center of mass of the wire. The wire is bent into the shape of the quarter circle C given by

$$x = 2\cos t\ ,\ y = 2\sin t\ ,\ z = 3\ ,\ \text{for } 0 \le t \le \frac{\pi}{2}$$

The density function is $\delta(x\,,\,y\,,\,z) = xy^2$ grams/cm^3. (15%)

《喻超凡, 喻超弘 102中興精密》

《解》 ☞ C 的位置向量爲

$$\vec{r} = x\,\vec{i} + y\,\vec{j} = 2\cos t\,\vec{i} + 2\sin t\,\vec{j}$$

故

$$ds = |\frac{d\vec{r}}{dt}|dt = 2\,dt$$

其中 $\mathscr{L}\{q(t)\} = Q(s)$, 整理可得

$$(s^2 + 20s + 25)Q(s) = 1$$

即

$$Q(s) = \frac{1}{s^2 + 20s + 25} = \frac{1}{(s+10)^2 - 75}$$

故

$$q(t) = \mathscr{L}^{-1}\{Q(s)\} = e^{-10t}\frac{1}{\sqrt{75}}\sinh(\sqrt{75}t)$$

3. The point $A(1\,,\,-2\,,\,1)$, $B(0\,,\,1\,,\,6)$, and $C(-3\,,\,4\,,\,-2)$ from the vertices of a triangle. Please calculate the cosine of the angle between AB and BC. (15%)

《喻超凡, 喻超弘 102中興精密》

《解》☞ $\overrightarrow{AB} = (-1\,,\,3\,,\,5)$ 、$\overrightarrow{BC} = (-3\,,\,3\,,\,-8)$, 故

$$\cos\theta = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}\cdot\frac{\overrightarrow{BC}}{|\overrightarrow{BC}|} = \frac{3+9-40}{\sqrt{35}\,\sqrt{82}} = -\frac{28}{\sqrt{2870}}$$

4. Please find the mass and center of mass of the wire. The wire is bent into the shape of the quarter circle C given by

$$x = 2\cos t\ ,\ y = \sin t\ ,\ z = 3\ ,\ \text{for } 0 \le t \le \frac{\pi}{2}$$

The density function is $\delta(x\,,\,y\,,\,z) = xy^2$ grams/cm^3. (15%)

《喻超凡, 喻超弘 102中興精密》

《解》☞ C 的位置向量爲

$$\vec{r} = x\vec{i} + y\vec{j} = 2\cos t\,\vec{i} + 2\sin t\,\vec{j}$$

故

$$ds = |\frac{d\vec{r}}{dt}|dt = 2\,dt$$

則 C 的質量爲

$$m=\int_C \delta(x\,,\,y\,,\,z)\,ds=\int_C xy^2\,ds=\int_0^{\frac{\pi}{2}}(2\cos t)(2\sin t)^2\;2\,dt=\frac{16}{3}\sin^3 t\Big|_0^{\frac{\pi}{2}}=\frac{16}{3}$$

$$\begin{aligned}M_x &= \int_C x\delta(x\,,\,y\,,\,z)\,ds=\int_C x^2y^2\,ds\\ &= \int_0^{\frac{\pi}{2}}(2\cos t)^2(2\sin t)^2\;2\,dt=16B(\frac{3}{2}\,,\,\frac{3}{2})\\ &= 16\frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2})}{\Gamma(3)}=16\,\frac{\frac{1}{2}\sqrt{\pi}\;\frac{1}{2}\sqrt{\pi}}{2}=2\pi\end{aligned}$$

$$M_y=\int_C y\delta(x\,,\,y\,,\,z)\,ds=\int_C xy^3\,ds=\int_0^{\frac{\pi}{2}}(2\cos t)(2\sin t)^3\;2\,dt=\frac{32}{4}\sin^4 t\Big|_0^{\frac{\pi}{2}}=8$$

設質心爲 $(\overline{x}\,,\,\overline{y}\,,\,\overline{z})$, 則

$$\overline{x}=\frac{M_x}{m}=\frac{3\pi}{8}\;,\;\overline{y}=\frac{M_y}{m}=\frac{3}{2}\;,\;\overline{z}=3$$

5. This is non-autonomous, where f and g have explicit $t-$ dependencies, Please solve the system. (15%)

$$x'(t)=\frac{1}{t}\,x=f(t\,,\,x\,,\,y)\;,\;y'=-\frac{1}{t}y+x=g(t\,,\,x\,,\,y)$$

《喻超凡, 喻超弘 102中興精密》

《解》 ☞ 由

$$x'(t)=\frac{1}{t}\,x\;\Rightarrow\;\frac{dx}{x}=\frac{1}{t}\,dt$$

兩端積分, 可得

$$\ln|x|=\ln|t|+c_1^*\;\Rightarrow\;x=c_1t$$

再由

$$y'=-\frac{1}{t}y+x=-\frac{1}{t}y+c_1t$$

整理可得

$$y'+\frac{1}{t}y=c_1t$$

積分因子 $I=\exp\{\int \frac{1}{t}\,dt\}=\exp\{\ln|t|\}=t$, 故

$$Iy=\int t\,(c_1 t)\,dt=c_1\frac{t^3}{3}+c_2$$

則

$$y(t)=c_1\,\frac{t^2}{3}+c_2\,\frac{1}{t}$$

6. Imaging a particle moving along a path having position vector

$$\vec{F}(t)=\sin t\,\vec{i}+2e^t\,\vec{j}+t^2\,\vec{k}$$

Please write the velocity, acceleration, and speed of the particle. (15%)
《喻超凡, 喻超弘 102中興精密》

《解》 ☞ velocity 爲

$$\vec{V}=\frac{d\vec{F}}{dt}=\cos t\,\vec{i}+2e^t\,\vec{j}+2t\,\vec{k}$$

acceleration

$$\vec{a}=\frac{d^2\vec{F}}{dt^2}=-\sin t\,\vec{i}+2e^t\,\vec{j}+2\,\vec{k}$$

speed 爲

$$|\vec{V}|=\sqrt{\cos^2 t+4e^{2t}+4t^2}$$

7. Solve the following differential equation. $y''+4y'+3y=e^t$, $y(0)=0$, $y'(0)=2$. (10%)
《喻超凡, 喻超弘 102中興精密》

《解》 ☞

(1) 齊次解：令 $y=e^{mt}$ 代回 ODE 中可得

$$m^2+4m+3=0\ \Rightarrow\ m=-1\,,-3$$

則

$$u_h(t) = e^{-\alpha t}(c_1 \cos wt + c_2 \sin wt)$$

(c) $c^2 - 4mk = 0$ 臨界阻尼 (critical damping)

特徵方程式的根爲重根，令

$$\lambda_1 = \lambda_2 = -\alpha = -\frac{c}{2m}$$

則

$$u_h(t) = e^{-\alpha t}(c_1 + c_2 t)$$

(2) 特解

$$\begin{aligned} u_p(t) &= \frac{1}{mD^2 + cD + k} \sin wt = \frac{1}{m(-w^2) + cD + k} \sin wt \\ &= \frac{cD - (k - mw^2)}{\{cD + (k - mw^2)\}\{cD - (k - mw^2)\}} \sin wt \\ &= \frac{cD - (k - mw^2)}{c^2D^2 - (k - mw^2)^2} \sin wt \\ &= \frac{1}{-c^2w^2 - (k - mw^2)^2} \{cw \cos wt - (k - mw^2) \sin wt\} \end{aligned}$$

(3) 通解爲：$u(t) = u_h(t) + u_p(t)$

2. (20%) Please find the principle stresses (Eigenvalue) and their orientation (Eigenvectors) given $\sigma_x = 60$, $\sigma_y = 100$, $\tau_{xy} = 20$.

《喻超凡, 喻超弘 102中興土木丙》

《解》 ☞ 令

$$A = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \begin{bmatrix} 60 & 20 \\ 20 & 100 \end{bmatrix}$$

由

$$\det(A - \lambda I) = \lambda^2 - 160\lambda + 5600 = 0$$

故 A 的特徵值爲 $\lambda = 80 \pm 20\sqrt{2}$，將 $\lambda = 80 + 20\sqrt{2}$ 代回 $(A - \lambda I)X = 0$ 中可得

$$\begin{bmatrix} -20 - 20\sqrt{2} & 20 \\ 20 & 20 - 20\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

對應的特徵向量為

$$X_1 = c_1 \begin{bmatrix} 1 \\ 1+\sqrt{2} \end{bmatrix} \quad (c_1 \neq 0)$$

將 $\lambda = 80 - 20\sqrt{2}$ 代回 $(A - \lambda I)X = 0$ 中可得

$$\begin{bmatrix} -20+20\sqrt{2} & 20 \\ 20 & 20+20\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

對應的特徵向量為

$$X_2 = c_2 \begin{bmatrix} 1 \\ 1-\sqrt{2} \end{bmatrix} \quad (c_2 \neq 0)$$

故主應為 $80+20\sqrt{2}$ 時，主軸為 X_1，主應為 $80-20\sqrt{2}$ 時，主軸為 X_2

3. (20%) Find Fourier transform of $f(x)$

$$f(x) = \begin{cases} A & ; |x| < t_1 \\ 0 & ; |x| \geq t_1 \end{cases}$$

《喻超凡, 喻超弘 102中興土木丙》

《解》☞

$$\begin{aligned} \mathscr{F}\{f(x)\} &= \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx \\ &= \int_{-t_1}^{t_1} A(\cos wx - i \sin wx)\, dx \\ &= 2\int_0^{t_1} A\cos wx\, dx = 2A \frac{\sin wx}{w}\Big|_0^{t_1} \\ &= \frac{2A\, \sin wt_1}{w} \end{aligned}$$

4. Please solve the governing equation of Terzaghi consolidation theory

$$c_v \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t}$$

Initial condition $u(z\,,\,0) = p$, Boundary condition $u(0\,,\,t) = 0$, $u(2H\,,\,t) = 0$.

《喻超凡, 喻超弘 102中興土木丙》

《解》 ☞ 因 $u(0\,,\,t)=u(2H\,,\,t)=0$, 故令

$$u(z\,,\,t)=\sum_{n=1}^{\infty}a_n(t)\sin\frac{n\pi z}{2H}$$

代回 PDE 中可得

$$\sum_{n=1}^{\infty}a_n'(t)\sin\frac{n\pi x}{2H}=c_v\sum_{n=1}^{\infty}a_n(t)(-\frac{n^2\pi^2}{4H^2})\sin\frac{n\pi z}{2H}$$

即

$$\sum_{n=1}^{\infty}\{a_n'(t)+c_v(\frac{n\pi}{2H})^2a_n(t)\}\sin\frac{n\pi z}{2H}=0$$

故

$$a_n'(t)+c_v(\frac{n\pi}{2H})^2a_n(t)=0$$

其解爲 $a_n(t)=A_ne^{-c_v(\frac{n\pi}{2H})^2t}$, 故

$$u(z\,,\,t)=\sum_{n=1}^{\infty}A_ne^{-c_v(\frac{n\pi}{2H})^2t}\sin\frac{n\pi z}{2H}$$

又

$$u(z\,,\,0)=p=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi z}{2H}$$

且

$$A_n=\frac{2}{2H}\int_0^{2H}p\sin\frac{n\pi z}{2H}\,dz=\frac{2p(1-\cos n\pi)}{n\pi}$$

則

$$u(z\,,\,t)=\sum_{n=1}^{\infty}\frac{2p(1-\cos n\pi)}{n\pi}e^{-c_v(\frac{n\pi}{2H})^2t}\sin\frac{n\pi z}{2H}$$

5. Ultimate bearing capacity (q_0) can be expressed as

$$q_u=c\,N_c+q\,N_q+0.5B\,N_r$$

where $N_c=\dfrac{N_q-1}{\tan\psi}$, and $N_q=\tan^2(\frac{\pi}{4}+\frac{\psi}{2})e^{-\pi\tan\psi}$. Please find N_c given $\psi=0$.

《喻超凡, 喻超弘102中興土木丙》

4. (20%) Use the Fourier series method to solve the problem :

$$u_t = 4u_{xx} \quad 0 < x < 2\ ,\ t > 0$$

$u(0\,,\,t) = u(2\,,\,t) = 0\ ,\ t > 0\ ;\ u(x\,,\,0) = 2\{1 - \cos(4\pi x)\}\ ,\ 0 < x < 2.$

《喻超凡, 喻超弘 102成大航太》

《解》☞ 因 $u(0\,,\,t) = u(2\,,\,t) = 0$, 故令

$$u(x\,,\,t) = \sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi x}{2}$$

代回 PDE 中可得

$$\sum_{n=1}^{\infty} a_n'(t) \sin\frac{n\pi x}{2} = 4\sum_{n=1}^{\infty} a_n(x)(-\frac{n^2\pi^2}{2^2}) \sin\frac{n\pi x}{2}$$

即

$$\sum_{n=1}^{\infty} \{a_n'(t) + (n\pi)^2 a_n(t)\} \sin\frac{n\pi x}{2} = 0$$

即

$$a_n'(t) + (n\pi)^2 a_n(t) = 0$$

故 $a_n(t) = A_n e^{-(n\pi)^2 t}$, 則

$$u(x\,,\,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 t} \sin\frac{n\pi x}{2}$$

再由

$$u(x\,,\,0) = 2\{1 - \cos(4\pi x)\} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{2}$$

故

$$A_n = \frac{2}{2}\int_0^2 2\{1 - \cos(4\pi x)\} \sin\frac{n\pi x}{2}\, dx = \frac{256(-1 + \cos n\pi)}{(-64n + n^3)\pi} \quad (n \neq 8)$$

$$A_8 = \frac{2}{2}\int_0^2 2\{1 - \cos(4\pi x)\} \sin\frac{8\pi x}{2}\, dx = 0$$

即

$$u(x\,,\,t) = \sum_{\substack{n=1 \\ n\neq 8}}^{\infty} \frac{256(-1 + \cos n\pi)}{(-64n + n^3)\pi} e^{-(n\pi)^2 t} \sin\frac{n\pi x}{2}$$

5. (20%) Consider the function of complex variable

$$f(z)=\frac{e^{az}}{e^{z}+1}\ ,\ 0<a<1$$

(a) Locate the singularities and evaluate the residues of $f(z)$.

(b) Evaluate the following integral using the Residue Theorem

$$I=\int_{-\infty}^{\infty}\frac{e^{ax}}{e^{x}+1}dx\ ,\ 0<a<1$$

《喻超凡, 喻超弘 102成大航太》

《解》☞

(a) 因 $e^z+1=0$, 可得

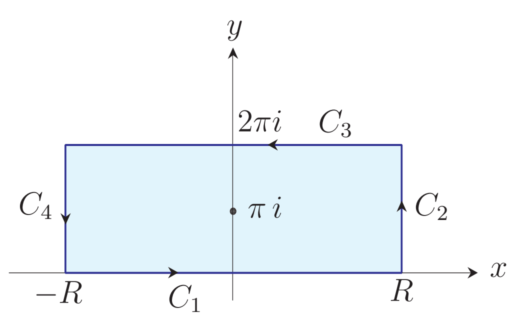
$$e^z=-1=e^{(2n\pi+\pi)i}\quad (n=0\,,\pm1\,,\pm2\,,\cdots)$$

即

$$z=(2n\pi+\pi)i\quad (n=0\,,\pm1\,,\pm2\,,\cdots)$$

爲 $f(z)$ 的一階 poles , 且

$$\text{Res}f((2n\pi+\pi)i)=\frac{e^{a(2n\pi+\pi)i}}{e^{(2n\pi+\pi)i}}=-e^{a(2n\pi+\pi)i}\quad (n=0\,,\pm1\,,\pm2\,,\cdots)$$



(b) 因 $f(z)=\dfrac{e^{az}}{1+e^{z}}$, 故 $f(z)$ 在 C ($C=C_1+C_2+C_3+C_4$ 如圖) 內具有 $z=\pi i$ 的一階 pole , 且

$$\text{Res}f(\pi i)=\lim_{z\to\pi i}\frac{e^{az}}{e^{z}}=e^{(a-1)\pi i}=-e^{a\pi i}$$

《解》☞ 對 ODE 兩端取 L–T，可得

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = 1 - e^{-\pi s}$$

其中 $\mathscr{L}\{y(t)\} = Y(s)$，整理可得

$$Y(s) = \frac{1}{s^2+4} - \frac{1}{s^2+4}e^{-\pi s}$$

故

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \frac{1}{2}\sin 2t - \frac{1}{2}\sin 2(t-\pi)\,H(t-\pi)$$

4. (20%) Let S be a piecewise smooth closed surface bounding a region V. Show that volume of $V = \frac{1}{3}\iint_S (\vec{r}\cdot\vec{n})\,dS$. Where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

《喻超凡, 喻超弘 102成大土木》

《解》☞ 因

$$\iint_S (\vec{r}\cdot\vec{n})\,dS = \iiint_V (\nabla\cdot\vec{r})\,dV = \iiint_V 3\,dV$$

故

$$V = \iiint_V dV = \frac{1}{3}\iint_S (\vec{r}\cdot\vec{n})\,dS$$

5. (20%) For a circular membrane subject to a pressure $p(x, y) = x$, the mathematic model is

$$\nabla^2 u = x\ ,\quad u = 0 \text{ for } x^2 + y^2 = a$$

Find the solution $u(x, y)$. 《喻超凡, 喻超弘 102成大土木》

《解》☞ 因區域為圓的區域，故 $u(r, \theta) = u(r, \theta + 2\pi)$，則特徵函數為

$$\left\{1\ ,\ \cos n\theta\ ,\ \sin n\theta\right\}_{n=1}^{\infty}$$

因此由特徵函數展開法知，令

$$u(r, \theta) = a_0(r) + \sum_{n=1}^{\infty}\{a_n(r)\cos n\theta + b_n(r)\sin n\theta\}$$

34. 102 中山大學機電工程研究所

1. (10%) Find all solution or indicate the no solution exists.

$$\begin{cases} 4y + z = 0 \\ 12x - 5y - 3z = 34 \\ -6x + 4z = 8 \end{cases}$$

《喻超凡, 喻超弘 102中山機電》

《解》☞ 原式可改寫成

$$\begin{bmatrix} 0 & 4 & 1 \\ 12 & -5 & -3 \\ -6 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 34 \\ 8 \end{bmatrix}$$

令

$$A = \begin{bmatrix} 0 & 4 & 1 \\ 12 & -5 & -3 \\ -6 & 0 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 34 \\ 8 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 0 & 4 & 1 & 0 \\ 12 & -5 & -3 & 34 \\ -6 & 0 & 4 & 8 \end{array}\right] \xrightarrow{R_{32}^{(2)}} \left[\begin{array}{ccc|c} 0 & 4 & 1 & 0 \\ 0 & -5 & 5 & 50 \\ -6 & 0 & 4 & 8 \end{array}\right]$$

$$\xrightarrow{R_2^{(1/5)}} \left[\begin{array}{ccc|c} 0 & 4 & 1 & 0 \\ 0 & -1 & 1 & 10 \\ -6 & 0 & 4 & 8 \end{array}\right] \xrightarrow{R_{21}^{(4)}} \left[\begin{array}{ccc|c} 0 & 0 & 5 & 40 \\ 0 & -1 & 1 & 10 \\ -6 & 0 & 4 & 8 \end{array}\right]$$

故可解得 $z = 8$、$y = -2$、$x = 4$

2. (15%) Find the eigenvalues and the corresponding eigenvectors. Use the given λ.

$$A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix}, \ \lambda = 4$$

《喻超凡, 喻超弘 102中山機電》

《解》☞ 因

$$\nabla T=\frac{\partial T}{\partial x}\vec{i}+\frac{\partial T}{\partial y}\vec{j}+\frac{\partial T}{\partial z}\vec{k}=-\frac{2xz}{(x^2+y^2)^2}\vec{i}-\frac{2yz}{(x^2+y^2)^2}\vec{j}+\frac{1}{x^2+y^2}\vec{k}$$

故

$$\nabla T(P)=-4\vec{j}+\vec{k}$$

故熱流的方向為 $-\dfrac{\nabla T(P)}{|\nabla T(P)|}=-\dfrac{-4\vec{j}+\vec{k}}{\sqrt{17}}$

---

4. Solve the following ordinary differential equations (ODEs) :
   (a) (10%) $(1-2x-x^2)y''+2(1+x)y'-2y=0$

   (b) (13%) $2xy''+(1+x)y'+y=0$

   (c) (12%) $y''+3y'+2y=\begin{cases}0 & \text{, if } t<1\\ 1 & \text{, if } 1<t<2\\ 0 & \text{, if } t>2\end{cases}$ with I.C. $\begin{cases}y(1)=1\\ y'(1)=-1\end{cases}$

《喻超凡, 喻超弘 102中山機電》

《解》☞

(a) 已知一解為 $y_1=(1+x)$，故 ODE 另外一解為

$$\begin{aligned}
y_2&=(1+x)\int\frac{e^{-\int\frac{2(1+x)}{(1-2x-x^2)}dx}}{(1+x)^2}dx\\
&=(1+x)\int\frac{e^{\ln|x^2+2x-1|}}{(1+x)^2}dx\\
&=(1+x)\int\frac{x^2+2x-1}{(1+x)^2}dx\\
&=(1+x)\int\{1-\frac{2}{(1+x)^2}\}dx\\
&=(1+x)\{x+\frac{2}{1+x}\}\\
&=x(x+1)+2=x^2+x+2
\end{aligned}$$

故 ODE 的通解為

$$y(x)=c_1(x+1)+c_2(x^2+x+2)$$

(b) 原式可改寫成

$$(2xy')'-2y'+(1+x)y'+y=0$$

即

$$(2xy')' + (x-1)y' + y = 0 \;\Rightarrow\; (2xy')' + \{(x-1)y\}' = 0$$

兩端積分可得

$$2xy' + (x-1)y = c_1$$

即

$$y' + \frac{x-1}{2x}y = \frac{c_1}{2x}$$

積分因子爲

$$I = \exp\{\int \frac{x-1}{2x}\,dx\} = \exp\{\frac{x}{2} - \frac{1}{2}\ln|x|\} = \frac{e^{\frac{x}{2}}}{\sqrt{x}}$$

則

$$Iy = \int \frac{c_1}{2x}\frac{e^{\frac{x}{2}}}{\sqrt{x}}\,dx + c_2$$

故 ODE 的通解爲

$$y(x) = \frac{c_1}{2}\sqrt{x}\,e^{\frac{-x}{2}}\int \frac{e^{\frac{x}{2}}}{x\sqrt{x}}\,dx + c_2\sqrt{x}\,e^{\frac{-x}{2}}$$

(c) 令 $x = t-1$，故 $t = 1$ 時 $x = 0$，且

$$f(t) = f(x) = \begin{cases} 0 & ,\text{ if } x < 0 \\ 1 & ,\text{ if } 0 < x < 1 \\ 0 & ,\text{ if } x > 1 \end{cases} = H(x) - H(x-1)$$

且 ODE 可改寫成

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = f(x)\,,\;\; y(0) = 1\,,\;\; \frac{dy}{dx}(0) = -1$$

故 $\mathscr{L}\{f(x)\} = \frac{1}{s}e^{-0s} - \frac{1}{s}e^{-s}$，對 ODE 兩端取 L–T，可得

$$s^2Y(s) - sy(0) - \frac{dy}{dx}(0) + 3\{sY(s) - y(0)\} + 2Y(s) = \frac{1}{s}e^{-0s} - \frac{1}{s}e^{-s}$$

其中 $\mathscr{L}\{y(x)\} = Y(s)$, 整理可得

$$(s^2 + 3s + 2)Y(s) = s + 2 + \frac{1}{s}e^{-0s} - \frac{1}{s}e^{-s}$$

故

$$\begin{aligned} Y(s) &= \frac{1}{s+1} + \frac{1}{s(s+2)(s+1)}e^{-0s} - \frac{1}{s(s+2)(s+1)}e^{-s} \\ &= \frac{1}{s+1} + \{\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\}e^{-0s} - \{\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\}e^{-s} \end{aligned}$$

即

$$
\begin{aligned}
y(x) &= \mathscr{L}^{-1}\{Y(s)\} \\
&= e^{-x} + \{\frac{1}{2} - e^{-x} + \frac{1}{2}e^{-2x}\}H(x) - \{\frac{1}{2} - e^{-(x-1)} + \frac{1}{2}e^{-2(x-1)}\}H(x-1)
\end{aligned}
$$

則

$$
y(t) = e^{-(t-1)} + \{\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\}H(t-1) - \{\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\}H(t-2)
$$

5. Please answer the following questions :
   (a) (5%) What is "Fourier series" used for ?
   (b) (5%) What is "orthogonality" regarding the Fourier series ?
   (c) (10%) Please find the Fourier series according to the following periodic rectangular wave.

$$
F(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < 2 \end{cases} \quad , \ k \text{ isaconstant}
$$

《喻超凡, 喻超弘 102中山機電》

《解》 ☞

(a) Fourier 級數可用來表示任一週期函數，無論此週期函數是否連續。

(b) Fourier 級數它是 sine 與 cosine 正交函數所構成的無窮級數。

(c) 因 $F(x)$ 爲週期 4 的偶函數，故

$$
F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}
$$

其中

$$
a_0 = \frac{1}{2}\int_0^2 F(x)\,dx = \frac{1}{2}\int_0^1 k\,dx = \frac{k}{2}
$$

$$
a_n = \frac{2}{2}\int_0^2 F(x)\cos\frac{n\pi x}{2}\,dx = \int_0^1 k\,\cos\frac{n\pi x}{2}\,dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2}
$$

則 $x=\cos t$、$y=\sin t$，故速度

$$\vec{v}=\frac{d\vec{r}}{dt}=-\sin t\,\vec{i}+2\cos t\,\vec{j}$$

加速度

$$\vec{a}=\frac{d^2\vec{r}}{dt^2}=-\cos t\,\vec{i}-2\sin t\,\vec{j}$$

speed 爲

$$|\vec{v}|=\sqrt{(-\sin t)^2+(2\cos t)^2}=\sqrt{1+3\cos^2 t}$$

當 $t=2\pi$ 即在點 $(1,0)$ 時，speed 爲最大，且

$$|\vec{a}|=\sqrt{(-\cos t)^2+(-2\sin t)^2}=\sqrt{1+3\sin^2 t}$$

當 $t=\frac{\pi}{2}$ 即在點 $(0,2)$ 時，加速度爲最大，且

$$\vec{a}_t=(\vec{a}\cdot\frac{\vec{v}}{|\vec{v}|})\frac{\vec{v}}{|\vec{v}|}=\frac{-3\sin t\cos t}{1+3\cos^2 t}(-\sin t\,\vec{i}+2\cos t\,\vec{j})$$

5. (15%) Find the Fourier series for a given function $f(x)$ with intervals specified:

$f(x)=x$，$-\pi<x<\pi$，$f(x+2k\pi)=f(x)$，$-\infty<x<\infty$ and $k=\pm$integer

《喻超凡, 喻超弘 102中山海環》

《解》☞ 因 $f(x)$ 爲奇函數，故

$$f(x)=\sum_{n=1}^{\infty}b_n\sin nx$$

其中

$$b_n=\frac{2}{\pi}\int_0^{\pi}x\sin nx\,dx=\frac{-2\cos n\pi}{n}$$

6. (10%) Evaluate the improper integral $\int_0^{\infty}\frac{dx}{1+x^4}$.

with four simple poles at $z_1=e^{\frac{\pi i}{4}}$, $z_2=e^{\frac{3\pi i}{4}}$, $z_3=e^{\frac{5\pi i}{4}}$, $z_4=e^{\frac{7\pi i}{4}}$, on a full circle.

《喻超凡, 喻超弘 102中山海環》

可得對應的特徵向量為

$$X_2 = c_2 \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} \quad (c_2 \neq 0)$$

將 $\lambda = -\frac{2}{3}$ 代回 $(A - \lambda I)X = 0$ 中可得

$$\begin{bmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ 1 & \frac{1}{2} & 0 \\ -\frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

可得對應的特徵向量為

$$X_3 = c_3 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \quad (c_3 \neq 0)$$

(3) 令

$$P = \begin{bmatrix} 1 & 2 & 1 \\ 6 & 3 & -2 \\ -13 & -2 & -1 \end{bmatrix}$$

則

$$\begin{aligned} \lim_{n\to\infty} \sin^n(A\pi) &= \lim_{n\to\infty} P \begin{bmatrix} \sin^n(0\pi) & 0 & 0 \\ 0 & \sin^n(\frac{1}{2}\pi) & 0 \\ 0 & 0 & \sin^n(-\frac{2}{3}\pi) \end{bmatrix} P^{-1} \\ &= \begin{bmatrix} 1 & 2 & 1 \\ 6 & 3 & -2 \\ -13 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 6 & 3 & -2 \\ -13 & -2 & -1 \end{bmatrix}^{-1} \\ &= \frac{1}{84} \begin{bmatrix} 64 & 24 & 16 \\ 96 & 36 & 24 \\ -64 & -24 & -16 \end{bmatrix} \end{aligned}$$

(2) $f(x)$ 在 $[0\,,\,1]$ 的 Fourier sine 級數爲

$$f(x)=\sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

其中

$$b_n=\frac{2}{1}\int_0^1 x\,\sin(n\pi x)\,dx=\frac{-2\cos n\pi}{n\pi}$$

故

$$f(x)=\sum_{n=1}^{\infty}\frac{-2\cos n\pi}{n\pi}\sin(n\pi x)$$

$f(x)$ 在 $[0\,,\,1]$ 的 Fourier cosine 級數爲

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n\cos(n\pi x)$$

其中

$$a_0=\int_0^1 x\,dx=\frac{1}{2}$$

$$a_n=\frac{2}{1}\int_0^1 x\,\cos(n\pi x)\,dx=\frac{2(-1+\cos n\pi)}{n^2\pi^2}$$

故

$$f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{2(-1+\cos n\pi)}{n^2\pi^2}\cos(n\pi x)$$

FCS 收斂最快, 因 $f(x)$ 在 $[0\,,\,1]$ 展的 Fourier cosine 級數時，爲連續函數。FSS 及 FS 在展級數時，$f(x)$ 爲片斷連續。

6. (20%) Solve the following Laplace equation with Boundary Conditions.

PDE: $u_{xx}+u_{yy}=0$ or $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$, $0<x<L$, $0<y<\ell$

BCs: $u(0\,,\,y)=0$, $u_x(L\,,\,y)=0$ or $\dfrac{\partial u}{\partial x}(L\,,\,y)=0$,

$u(x\,,\,0)=0$, $u(x\,,\,\ell)=1$ 《喻超凡, 喻超弘 102海洋機電》

《解》☞ 因 $u(0\,,\,y)=u_x(L\,,\,y)=0$，故令

$$u(x\,,\,y)=\sum_{n=1}^{\infty} a_n(y)\sin\frac{(2n-1)\pi x}{2L}$$

代回 ODE 中可得

$$\sum_{n=1}^{\infty} a_n(y)\{-\frac{(2n-1)^2\pi^2}{4L^2}\}\sin\frac{(2n-1)\pi x}{2L}+\sum_{n=1}^{\infty} a_n''(y)\sin\frac{(2n-1)\pi x}{2L}=0$$

即

$$\sum_{n=1}^{\infty}\{a_n''(y)-\{\frac{(2n-1)\pi}{2L}\}^2 a_n(y)\}\sin\frac{(2n-1)\pi x}{2L}=0$$

可得

$$a_n''(y)-\{\frac{(2n-1)\pi}{2L}\}^2 a_n(y)=0$$

則

$$a_n(y)=A_n\sinh\frac{(2n-1)\pi y}{2L}+B_n\cosh\frac{(2n-1)\pi y}{2L}$$

PDE 的全解爲

$$u(x\,,\,y)=\sum_{n=1}^{\infty}\{A_n\sinh\frac{(2n-1)\pi y}{2L}+B_n\cosh\frac{(2n-1)\pi y}{2L}\}\sin\frac{(2n-1)\pi x}{2L} \tag{1}$$

由

$$u(x\,,\,0)=0=\sum_{n=1}^{\infty} B_n\sin\frac{(2n-1)\pi x}{2L}$$

故 $B_n=0$, 再由

$$u(x\,,\,\ell)=1=\sum_{n=1}^{\infty} A_n\sinh\frac{(2n-1)\pi\ell}{2L}\sin\frac{(2n-1)\pi x}{2L}$$

故

$$A_n\sinh\frac{(2n-1)\pi\ell}{2L}=\frac{2}{L}\int_0^L\sin\frac{(2n-1)\pi x}{2L}\,dx=\frac{4}{(2n-1)\pi}$$

將 A_n 、B_n 代回 (1) 式, 即爲所求 。

(3) 通解：

$$y(x) = y_h(x) + y_p(x) = c_1e^{-x} + c_2e^{x/2} - x - 2$$

再由

$$\begin{cases} y(0) = 1 = c_1 + c_2 - 2 \\ y'(0) = 0 = -c_1 + \frac{1}{2}c_2 - 1 \end{cases}$$

可解得 $c_1 = \frac{1}{3}$、$c_2 = \frac{8}{3}$，故

$$y(x) = \frac{1}{3}e^{-x} + \frac{8}{3}e^{x/2} - x - 2$$

3. (10%) Use the Laplace Transform to solve the given initial-value problem.

$$y'' + y = \sin t\,,\ y(0) = 1\,,\ y'(0) = -1$$

《**喻超凡, 喻超弘** 102海洋輪機》

《**解**》☞ 對 ODE 兩端取 L–T，可得

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s^2+1}$$

其中 $\mathscr{L}\{y(t)\} = Y(s)$, 整理可得

$$(s^2+1)Y(s) = \boxed{s - 1 + \frac{1}{s^2+1}}$$

即

$$Y(s) = \frac{s-1}{s^2+1} + \frac{1}{(s^2+1)^2}$$

因 $\mathscr{L}^{-1}\{\frac{1}{s^2+a^2}\} = \frac{1}{a}\sin at$, 兩端對 a 微分，可得

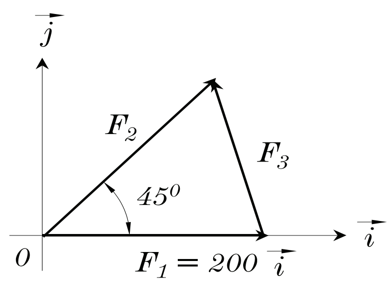
$$\mathscr{L}^{-1}\{\frac{-2a}{(s^2+a^2)^2}\} = -\frac{1}{a^2}\sin at + \frac{t}{a}\cos at$$

故

$$\mathscr{L}^{-1}\{\frac{1}{(s^2+a^2)^2}\} = \frac{1}{2a^3}\sin at - \frac{t}{2a^2}\cos at$$

5. (10%) 水由消防水管噴出會承受水平力 F_1，大小 200lb，見圖1，救火員必須施加多少的力 F_3, 使得水管能夠朝向水平 45° 的方向？

《喻超凡, 喻超弘 102海洋輪機》



《解》☞ 由圖形可知，

$$\vec{F}_2 = 200\cos 45°\ \vec{i} + 200\sin 45°\ \vec{j} = 100\sqrt{2}\ \vec{i} + 100\sqrt{2}\ \vec{j}$$

則

$$\vec{F}_3 = \vec{F}_2 - \vec{F}_1 = 100\sqrt{2}\ \vec{i} + 100\sqrt{2}\ \vec{j} - 200\ \vec{i} = (100\sqrt{2} - 200)\ \vec{i} + 100\sqrt{2}\ \vec{j}$$

6. (10%) 給定矩陣A爲對稱，求出使得A對角化的正交矩陣P及對角矩陣D，其中 $D = P^T AP$。

$$A = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}$$

《喻超凡, 喻超弘 102海洋輪機》

《解》☞ 由 $\det(A-\lambda I) = 0$ 可得 A 的特徵值爲 $\lambda = 1$、8、-6, 將 $\lambda = 1$ 代回 $(A-\lambda I)X = 0$ 中可得

$$\begin{bmatrix} 0 & 0 & 7 \\ 0 & 0 & 0 \\ 7 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

可得對應的特徵向量爲

$$X = c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (c_1 \neq 0) \quad \text{取} \quad X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

將 $\lambda = 8$ 代回 $(A - \lambda I)X = 0$ 中可得

$$\begin{bmatrix} -7 & 0 & 7 \\ 0 & -7 & 0 \\ 7 & 0 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

可得對應的特徵向量爲

$$X = c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad (c_2 \neq 0) \quad \text{取} \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

將 $\lambda = -6$ 代回 $(A - \lambda I)X = 0$ 中可得

$$\begin{bmatrix} 7 & 0 & 7 \\ 0 & 7 & 0 \\ 7 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

可得對應的特徵向量爲

$$X = c_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad (c_3 \neq 0) \quad \text{取} \quad X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

令

$$P = [\frac{X_1}{\|X_1\|} \quad \frac{X_2}{\|X_2\|} \quad \frac{X_3}{\|X_3\|}] = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

故

$$P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

7. (10%) Find the curl and divergence of the given vector field. 求出給定向量場的旋度和散度。

$$\vec{F}(x\,,\,y\,,\,z) = (x-y)^3\vec{i} + e^{-yz}\vec{j} + xye^{2y}\vec{k}$$

《喻超凡, 喻超弘 102海洋輪機》

(3) 相異的特徵值對應的特徵函數相對權函數 $w = \dfrac{1}{1+x^2}$ 爲正交。

8. 試求解下列的邊界值問題 (18%)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}\ ,\ 0 < x < 1\ ,\ t > 0$$

邊界條件：$u(0\,,\,t) = \sin t\ ,\ u(1\,,\,t) = 0,\ t > 0$

初始條件：$u(x\,,\,0) = 0\ ,\ 0 < x < 1.$ 《**喻超凡, 喻超弘** 101台大生機》

《**解**》☞ 令 $u(x\,,\,t) = \phi(x\,,\,t) - x\sin t + \sin t$, 代回 ODE 中可得

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \phi}{\partial t} - x\cos t + \cos t$$

且

$$u(0\,,\,t) = \phi(0\,,\,t) + \sin t = \sin t \ \Rightarrow\ \phi(0\,,\,t) = 0$$

$$u(1\,,\,t) = \phi(1\,,\,t) - \sin t + \sin t = 0 \ \Rightarrow\ \phi(1\,,\,t) = 0$$

$$\boxed{u(x\,,\,0) = \phi(x\,,\,0) = 0}$$

因 $\phi(0\,,\,t) = \phi(1\,,\,t) = 0$, 故特徵函數爲

$$\left\{\sin n\pi x\right\}_{n=1}^{\infty}$$

故

$$\phi(x\,,\,t) = \sum_{n=1}^{\infty} a_n(t)\sin n\pi x$$

且

$$(-x+1) = \sum_{n=1}^{\infty} q_n \sin n\pi x$$

其中

$$q_n = 2\int_0^1 (-x+1)\sin n\pi x\,dx = \frac{2}{n\pi}$$

代回 PDE 中可得

$$\sum_{n=1}^{\infty} a_n(t)(-n^2\pi^2)\sin n\pi x = \sum_{n=1}^{\infty} a_n'(t)\sin n\pi x + \cos t\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin n\pi x$$

即

$$\sum_{n=1}^{\infty}\frac{4(-1)^n}{\pi(1-4n^2)}=1-\frac{2}{\pi}$$

故

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(4n^2-1)}=\frac{\pi}{4}(\frac{2}{\pi}-1)=\frac{1}{2}-\frac{\pi}{4}$$

6. 某一質點在空間中移動, 其位置向量爲

$$\vec{S}(t)=(t^3-t)\vec{i}+(2t^2+t)\vec{j}+3t\vec{k}$$

其中 $t\geq 0$ 是時間, 求此質量在 $t=1$ 沿切線方向之加速度向量及沿法線方向之加速度向量。(10%) 《喻超凡, 喻超弘 101 台科大營建》

《解》☞ 質點的速度爲

$$\vec{V}=\frac{d\vec{S}}{dt}=(3t^2-1)\vec{i}+(4t+1)\vec{j}+3\vec{k}$$

故加速度爲

$$\frac{d\vec{V}}{dt}=6t\vec{i}+4\vec{j}$$

因此質點在 $t=1$ 時的加速度爲

$$\vec{a}=\frac{d\vec{V}}{dt}\Big|_{t=1}=6\vec{i}+4\vec{j}$$

且質點 $t=1$ 時的單位切線向量爲

$$\vec{T}=\frac{\vec{V}}{|\vec{V}|}\Big|_{t=1}=\frac{2\vec{i}+5\vec{j}+3\vec{k}}{\sqrt{38}}$$

故質點在 $t=1$ 的切線加速度爲

$$\vec{a}_T=(\vec{a}\cdot\vec{T})\vec{T}=\frac{16}{19}(2\vec{i}+5\vec{j}+3\vec{k})$$

故質點在 $t=1$ 的法線加速度爲

$$\vec{a}_N=\vec{a}-\vec{a}_T=\frac{1}{19}(82\vec{i}-4\vec{j}-48\vec{k})$$

2. (10%) Consider a homogenous linear 2nd order ordinary differential equation (ODE)

$$y'' + p(x)y' + q(x)y = 0$$

if y_1 is one solution of this ODE, you world find a second independent solution y_2 by reduction of order method, that is, $y_2 = uy_1$. Prove

$$u(x) = \int U(x)\,dx \ , \ \text{and} \ \ U(x) = \frac{1}{y_1^2} e^{-\int p(x)\,dx}$$

《喻超凡, 喻超弘 101清大工科》

《解》☞ 令 $y(x) = y_1(x)v(x)$, 故

$$y'(x) = y_1'(x)v(x) + y_1(x)v'(x)$$

$$y''(x) = y_1''(x)v(x) + 2y_1'(x)v'(x) + y_1(x)v''(x)$$

上式代入 ODE 式中

$$y_1''(x)v(x) + 2y_1'(x)v'(x) + y_1(x)v''(x) + p(x)\{y_1'(x)v(x) + y_1(x)v'(x)\}$$
$$+q(x)y_1(x)v(x) = 0$$

整理可得

$$y_1(x)v''(x) + \{p(x)y_1(x) + 2y_1'(x)\}v'(x) + \{y_1''(x) + p(x)y_1'(x) + q(x)y_1(x)\}v(x) = 0 \quad (1)$$

因 $y_1(x)$ 為 ODE 的一解, 故

$$y_1''(x) + p(x)y_1'(x) + q(x)y_1(x) = 0$$

代回 (1) 式, 可得

$$y_1(x)v''(x) + \{p(x)y_1(x) + 2y_1'(x)\}v'(x) = 0$$

分離變數可得

$$\frac{dv'(x)}{v'(x)} = -\frac{p(x)y_1(x) + 2y_1'(x)}{y_1(x)}\,dx$$

《解》☞ 令 $F(z)=\dfrac{1}{1+z^2}$, 故 $F(z)$ 在上半面具有 $z=i$ 的一階 pole, 且

$$\text{Res}F(i)=\frac{1}{2z}\Big|_{z=i}=\frac{1}{2i}$$

故

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx=2\pi\, i\,\text{Res}F(i)=\pi$$

7. 長度爲 L 的均勻圓棒, 未受外力時的運動方程式 (governing equation) 爲

$$\frac{\partial^2\theta}{\partial t^2}=\alpha^2\frac{\partial^2\theta}{\partial x^2}$$

其初始條件爲 $\theta(x\,,0)=kx$ 及 $\dfrac{\partial\theta}{\partial t}(x\,,0)=0$, 若此圓棒一端固定, 另一端自由 。

(a) (5%) 自由端的邊界條件可表示爲 $\dfrac{\partial\theta}{\partial x}(L\,,t)=0$, 請寫出固定端邊界條件的表示式 。

(b) (20%) 請解出 $\theta(x\,,t)$ 。 《喻超凡, 喻超弘 101交大機械甲》

《解》☞

(a) $\theta(0\,,t)=0$ 。

(b) 因 $\theta(0\,,t)=\dfrac{\partial\theta}{\partial x}(L\,,t)=0$, 故特徵函數爲

$$\Big\{\sin\frac{(2n-1)\pi x}{2L}\Big\}_{n=1}^{\infty}$$

令

$$\theta(x\,,t)=\sum_{n=1}^{\infty}a_n(t)\sin\frac{(2n-1)\pi x}{2L}$$

代回 ODE 中可得

$$\sum_{n=1}^{\infty}a_n''(t)\sin\frac{(2n-1)\pi x}{2L}=\alpha^2\sum_{n=1}^{\infty}a_n(t)\{-\frac{(2n-1)^2\pi^2}{4L^2}\}\sin\frac{(2n-1)\pi x}{2L}$$

整理可得

$$\sum_{n=1}^{\infty}\{a_n''(t)+(\frac{(2n-1)\pi\alpha}{2L})^2a_n(t)\}\sin\frac{(2n-1)\pi x}{2L}=0$$

將 $\lambda = 5$ 代回 $(A - \lambda I)X = 0$，中可得

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

對應的特徵向量爲

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

令

$$P = [X_1 \ \ X_2] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

故

$$P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

則

$$A^{301} = PD^{301}P^{-1} = P \begin{bmatrix} 1 & 0 \\ 0 & 5^{301} \end{bmatrix} P^{-1}$$

4. Solve the following differential equation by power series. (20%)

$$\frac{d^4y}{dx^4} + \sin x \frac{d^2y}{dx^2} = 0$$

Note : $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$.

《喻超凡, 喻超弘 101交大土木甲》

《解》☞ 因 $x = 0$ 爲 ODE 的常點, 故令

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n$$

令 $y(0) = c_1$、$y'(0) = c_2$、$y''(0) = c_3$、$y'''(0) = c_4$，再由

$$y^{(4)}(x) = -(\sin x)y''(x) \ \Rightarrow \ y^{(4)}(0) = 0$$

$$y^{(5)}(x) = -(\cos x)y''(x) - (\sin x)y'''(x) \ \Rightarrow \ y^{(5)}(0) = -y''(0) = -c_3$$

$$y^{(6)}(x) = (\sin x)y''(x) - 2(\cos x)y'''(x) - (\sin x)y^{(4)}(x) \ \Rightarrow \ y^{(6)}(0) = -2y'''(x) = -2c_4$$

$$\vdots$$