101-102

工程數學歷屆試題

電機所

勘誤表

可得對應的特徵向量為

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

將 $\lambda = 1$ 代回 $(A - \lambda I)X = 0$ 中可得

$$\begin{bmatrix} 2 & -2 & 0 \\ 1 & 2 & -3 \\ 1 & 4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

可得對應的特徵向量為

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

將 $\lambda = 2$ 代回 $(A - \lambda I)X = 0$ 中可得

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -3 \\ 1 & 4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

可得對應的特徵向量為

$$X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

因為 $\lambda_1 \geq \lambda_2 \geq \lambda_3$, 故

$$P = [X_3 \quad X_2 \quad X_1] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

且

$$\det(B) = \det(PM_2P^{-1}) = \det(P) \cdot \det(M_2) \cdot \det(P^{-1}) = \det(M_2) = 2$$

$$\begin{aligned} AB - BA &= PM_1P^{-1}PM_2P^{-1} - PM_2P^{-1}PM_1P^{-1} \\ &= PM_1M_2P^{-1} - PM_2M_1P^{-1} \\ &= PM_1M_2P^{-1} - PM_1M_2P^{-1} \\ &= 0 \end{aligned}$$

9. (5%) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors with norms $\|\boldsymbol{u}\| = 4$ and $\|\boldsymbol{v}\| = 7$, respectively, in some inner product space. Let $< 2\boldsymbol{u} + \boldsymbol{v}\ ,\ 2\boldsymbol{u} - \boldsymbol{v} > =$ (A14)
(A14) = ? (5%)　《喻超凡, 喻超弘 102台大電機 C》

《解》 ☞ 由題意可知 $B = AEc_{12}Ec_{35}$，且 A 的最簡列梯矩陣為

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

故存在一個可逆的矩陣 P 使得

$$PA = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

上式兩端後乘 $Ec_{12}Ec_{35}$，即

$$PAEc_{12}Ec_{35} = PB = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} Ec_{12}Ec_{35} = \begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

則

$$Er_{12}PB = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$Er_{32}^{(-1)}Er_{31}^{(1)}\boxed{Er_{12}PB} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

故 B 的最簡列梯矩陣為

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

13. (10%) Let T be a linear operator an $M_{2\times 2}$ and $T(A) = A - 2A^T$. Find an expression for

$$T^{-1}(\begin{bmatrix} a & b \\ c & d \end{bmatrix})$$

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《解》 ☞ 令

$$T^{-1}(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

中可得

$$\Phi'(t)Y(t)+\Phi(t)Y'(t)=A\Phi(t)Y(t)+B$$

即

$$Y'(t)=\Phi^{-1}(t)B \;\Rightarrow\; Y(t)=\int \Phi^{-1}(t)B\,dt$$

故

$$\begin{aligned}
X_p(t) &= \Phi(t)Y(t)\\
&= \Phi(t)\int \Phi^{-1}B\,dt\\
&= \Phi(t)\int \begin{bmatrix} e^{-2t} & e^{-5t}\\ e^{-2t} & -2e^{-5t}\end{bmatrix}^{-1}\begin{bmatrix} 3t\\ e^{-t}\end{bmatrix}dt\\
&= \begin{bmatrix} e^{-2t} & e^{-5t}\\ e^{-2t} & -2e^{-5t}\end{bmatrix}\begin{bmatrix} \frac{1}{3}e^{t}+\frac{1}{3}e^{2t}(t-\frac{1}{2})\\ -\frac{1}{12}e^{4t}+e^{5t}(\frac{t}{5}-\frac{1}{25})\end{bmatrix}\\
&= \begin{bmatrix} \frac{1}{4}e^{-t}+\frac{6}{5}t-\frac{27}{50}\\ \frac{1}{2}e^{-t}+\frac{3}{5}t-\frac{21}{50}\end{bmatrix}
\end{aligned}$$

(3) ODE 的通解爲

$$\begin{bmatrix} x(t)\\ y(t)\end{bmatrix}=X_h(t)+X_p(t)=\begin{bmatrix} e^{-2t} & e^{-5t}\\ e^{-2t} & -2e^{-5t}\end{bmatrix}\begin{bmatrix} c_1\\ c_2\end{bmatrix}+\begin{bmatrix} \frac{1}{4}e^{-t}+\frac{6}{5}t-\frac{27}{50}\\ \frac{1}{2}e^{-t}+\frac{3}{5}t-\frac{21}{50}\end{bmatrix}$$

再由

$$\begin{bmatrix} x(0)\\ y(0)\end{bmatrix}=\begin{bmatrix} 0\\ 0\end{bmatrix}=\begin{bmatrix} 1 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix} c_1\\ c_2\end{bmatrix}+\begin{bmatrix} \frac{1}{4}-\frac{27}{50}\\ \frac{1}{2}-\frac{21}{50}\end{bmatrix}$$

可解得 $c_1=\frac{1}{6}$、$c_2=\frac{37}{300}$，故

$$\begin{bmatrix} x(t)\\ y(t)\end{bmatrix}=\begin{bmatrix} e^{-2t} & e^{-5t}\\ e^{-2t} & -2e^{-5t}\end{bmatrix}\begin{bmatrix} \frac{1}{6}\\ \frac{37}{300}\end{bmatrix}+\boxed{\begin{bmatrix} \frac{1}{4}e^{-t}+\frac{6}{5}t-\frac{27}{50}\\ \frac{1}{2}e^{-t}+\frac{3}{5}t-\frac{21}{50}\end{bmatrix}}$$

(d) 因 $a=3$、$b=1$、$c=4$、$d=2$，故 ODE 可改寫成

$$\frac{d^2y}{dt^2}+7\frac{dy}{dt}+10y=2e^{-t}+6t\;,\;y(0)=0\;,\;y'(0)=1$$

可解得

$$\begin{cases} x_1(t) = c_1e^{-4t} - 2e^{-2t} \\ x_2(t) = c_2e^{-2t} - 2te^{-2t} \end{cases}$$

故

$$\begin{aligned} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= PX = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1e^{-4t} - 2e^{-2t} \\ c_2e^{-2t} - 2te^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} c_1e^{-4t} + c_2e^{-2t} - 2e^{-2t} - 2te^{-2t} \\ -c_1e^{-4t} + c_2e^{-2t} + 2e^{-2t} - 2te^{-2t} \end{bmatrix} \end{aligned}$$

4. 試求出下列行列式

(1) (5%) $D = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \\ 1 & 1 & 1 & \cdots & 0 \end{vmatrix}$ (2) (10%) $D = \begin{vmatrix} a_1 & x & x & \cdots & x \\ x & a_2 & x & \cdots & x \\ x & x & a_3 & \cdots & x \\ \cdots & \cdots & \cdots & \cdots & \\ x & x & x & \cdots & a_n \end{vmatrix}$

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《解》 ☞

(1) 因

$$\begin{aligned} D &= \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{n\times n} \\ &= \det \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{n\times n} + \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & 1 & 1 & \cdots & 0 \end{bmatrix}_{n\times n} \\ &= \det \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{n\times n} + \det \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{(n-1)\times(n-1)} \end{aligned}$$

$$= \det(A)+\det(B)$$

又矩陣 A 的特徵值爲 $\lambda=(n-1)$、(-1)、(-1)、$\cdots$、(-1) ($n-1$ 個) , 故

$$\det(A)=(-1)^{n-1}(n-1)$$

矩陣 B 的特徵值爲 $\lambda=(n-2)$、(-1)、(-1)、$\cdots$、(-1) ($n-2$ 個) , 故

$$\det(B)=(-1)^{n-2}(n-2)$$

則

$$\begin{aligned} D &= \det(A)+\det(B)=(-1)^{n-1}(n-1)+(-1)^{n-2}(n-2) \\ &= (-1)^{n-2}\{-(n-1)+(n-2)\}=(-1)^{n-1} \end{aligned}$$

(2)

$$D = \begin{vmatrix} a_1 & x & x & \cdots & x \\ x & a_2 & x & \cdots & x \\ x & x & a_3 & \cdots & x \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x & x & x & \cdots & a_n \end{vmatrix} \quad \text{(第1行乘 } (-1) \text{ 加到第2行到第}n\text{行)}$$

$$= \begin{vmatrix} a_1 & x & x & \cdots & x \\ x-a_1 & a_2-x & 0 & \cdots & 0 \\ x-a_1 & 0 & a_3-x & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x-a_1 & 0 & 0 & \cdots & a_n-x \end{vmatrix}$$

$$= (x-a_1)(a_2-x)\cdots(a_n-x)\begin{bmatrix} \dfrac{a_1}{x-a_1} & \dfrac{x}{a_2-x} & \dfrac{x}{a_3-x} & \cdots & \dfrac{x}{a_n-x} \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

(從第二行開始乘 (-1) 加到第一行)

$$= (x-a_1)(a_2-x)\cdots(a_n-x)\begin{bmatrix} \beta & \dfrac{x}{a_2-x} & \dfrac{x}{a_3-x} & \cdots & \dfrac{x}{a_n-x} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

10. Which of the following statements are true ?

(A) Let T be a linear transformation from $\mathbb{R}^n$ onto $\mathbb{R}^m$. Then T is one-to-one.

(B) Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k\}$ can be linearly dependent even if $\{T\boldsymbol{v}_1, \cdots, T\boldsymbol{v}_k\}$ is linearly independent.

(C) Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, and let $S \subset \mathbb{R}^n$ be a subspace such that $T\boldsymbol{s} \in S$ for all $\boldsymbol{s} \in S$. Then $\mathbf{dim}(S) \in \{0, 1, n\}$, where $\mathbf{dim}(S)$ denotes the dimension of S.

(D) Let $A \in \mathbb{R}^{m\times n}$ with $m > n > 3$. Then $\mathbf{rank}(AA^T) < \mathbf{rank}(A)$.

(E) None of the above are true. 《喻超凡, 喻超弘 102台聯 C》

《解》☞

(A) False :

(B) False ; $\{T\boldsymbol{v}_1, \cdots, T\boldsymbol{v}_k\}$ 爲線性獨立，則 $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k\}$ 一定線性獨立，但逆定理不恆眞 。

(C) False ;

(D) False ; 應該是 $\mathbf{rank}(AA^T) = \mathbf{rank}(A)$.

(E) True ;

可得 $X(10)=0$ 或 $T(t)=0$，因 $T(t)=0$ 會使得 $g(x\,,\,t)=X(x)T(t)=0$，即方程式只有零解 (trivial 解)，無非零解，因此取 $X(10)=0$，同理

$$g(20\,,\,t)=0=X(20)T(t)$$

可得 $X(20)=0$，現解

$$X''(x)+\lambda X(x)=0\,,\;\; X(10)=X(20)=0 \tag{2}$$

Sturm–Liouville 邊界值問題

(i) $\lambda<0$，令 $\lambda=-p^2$ $(0<p<\infty)$ 代入 (2) 式，可得

$$X''(x)-p^2X(x)=0$$

其通解爲

$$X(x)=c_1\sinh p(x-10)+c_2\cosh p(x-10)$$

代入邊界條件 $X(10)=0$，可得 $c_2=0$，$X(20)=0=c_1\sinh 10p$，可得 $c_1=0$，故 $X(x)=0$，則 $g(x,t)=X(x)T(t)=0$ 無非零解。

(ii) $\lambda=0$，代入 (2) 式，可得 $X''(x)=0$，其通解爲

$$X(x)=c_1+c_2(x-10)$$

代入邊界條件 $X(10)=0$，可得 $c_1=0$，$X(20)=0=c_2\times 10$，可得 $c_2=0$，故 $X(x)=0$，則 $g(x,t)=X(x)T(t)=0$ 無非零解。

(iii) $\lambda>0$，令 $\lambda=p^2$ $(0<p<\infty)$ 代入 (2) 式，可得

$$X''(x)+p^2X(x)=0$$

其通解爲

$$X(x)=c_1\sin p(x-10)+c_2\cos p(x-10)$$

代入邊界條件 $X(10)=0$，可得 $c_2=0$，$X(20)=0=c_1\sin 10p$，令 $c_1\neq 0$，可得 $\sin 10p=0$，故

$$p=\frac{n\pi}{10}\quad (n=1\,,\,2\,,\,3\,,\,\cdots)$$

則

$$X(x)=c_1\sin\frac{n\pi(x-10)}{10}=c_1\sin(\frac{n\pi x}{10}-n\pi)=c_1(-1)^n\sin\frac{n\pi x}{10}=X_n$$

將 $\lambda=p^2=\boxed{(\frac{n\pi}{10})^2}$ 代回 (1) 式中可得

$$\dot{T}+5(\frac{n\pi}{10})^2T=0$$

上式爲一階 ODE, 其解爲

$$T(t) = de^{-(\frac{n^2\pi^2}{20})t} = T_n \quad (n = 1\,,\,2\,,\,3\,,\cdots)$$

因此

$$\begin{aligned} g_n(x\,,\,t) &= T_n(t)X_n(x) = de^{-(\frac{n^2\pi^2}{20})t}\,c_1(-1)^n \sin\frac{n\pi x}{10} \\ &= c_n\,e^{-(\frac{n^2\pi^2}{20})t}\,\sin\frac{n\pi x}{10} \end{aligned}$$

(3) 由重疊原理知

$$g(x\,,\,t) = \sum_{n=1}^{\infty} g_n(x,t) = \sum_{n=1}^{\infty} c_n\,e^{-(\frac{n^2\pi^2}{20})t}\,\sin\frac{n\pi x}{10}$$

(4) 本題答案選 (B)

20. Continued from Question 19, we further assume that $g(x\,,\,0) = x$ for $10 < x < 20$ and that the solution $g(x\,,\,t)$ takes the form of

$$g(x\,,\,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t)$$

for some constants c_n. Which of the following statements are true ?

(A) $c_n = \dfrac{20((-1)^n - 2)}{n\pi}$, $X_n(x) = \sin(\dfrac{n\pi}{10}x)$ and $T_n(t) = \exp(-\dfrac{n^2\pi^2}{20}t)$

(B) $c_n = \dfrac{20(1 - 2(-1)^n)}{n\pi}$, $X_n(x) = \sin(\dfrac{n\pi}{10}x)$ and $T_n(t) = \exp(-\dfrac{n^2\pi^2}{20}t)$

(C) $\displaystyle\sum_{n=1}^{\infty} c_n X_n(20) T_n(0) = 20$

(D) $\displaystyle\sum_{n=1}^{\infty} c_n X_n(10) T_n(0) = 0$

(E) None of the above are true. 《喻超凡, 喻超弘 102台聯 C》

《解》 ☞ 由

$$g(x\,,\,0) = x = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{10}$$

5. (25%) Let V be a subspace of $\mathbb{R}^5$ generated by

$$\left\{\begin{bmatrix}1\\3\\-3\\-1\\-4\end{bmatrix},\begin{bmatrix}1\\4\\-1\\-2\\-2\end{bmatrix},\begin{bmatrix}2\\9\\0\\-5\\-2\end{bmatrix}\right\}$$

and W be a subspace generated by

$$\left\{\begin{bmatrix}1\\6\\2\\-2\\3\end{bmatrix},\begin{bmatrix}2\\8\\-1\\-6\\-5\end{bmatrix},\begin{bmatrix}1\\3\\-1\\-5\\-6\end{bmatrix}\right\}$$

(1) (10%) Find a basis and dimension for $V+W$. You must justify your answer mathematically.

(2) (15%) Find a basis and dimension for $V\cap W$. You must justify your answer mathematically. 《喻超凡, 喻超弘 102台聯 D》

《解》☞ 令

$$A=\begin{bmatrix}1&1&2&1&2&1\\3&4&9&6&8&3\\-3&-1&0&2&-1&-1\\-1&-2&-5&-2&-6&-5\\-4&-2&-2&3&-5&-6\end{bmatrix}$$

$$\xrightarrow{R_{12}^{(-3)}R_{13}^{(3)}R_{14}^{(1)}R_{15}^{(4)}}\begin{bmatrix}1&1&2&1&\boxed{2\quad 1}\\0&1&3&3&2&0\\0&2&6&5&5&2\\0&-1&-3&-1&-4&-4\\0&2&6&7&3&-2\end{bmatrix}$$

即

$$\begin{cases} a+b+2c+(-d)+2(-e)+(-f)=0 \\ b+3c+3(-d)+2(-e)=0 \\ -(-d)+(-e)+2(-f)=0 \end{cases}$$

可解得

$$\begin{bmatrix} a \\ b \\ c \\ -d \\ -e \\ -f \end{bmatrix} = c\begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (-e)\begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + (-f)\begin{bmatrix} 3 \\ -6 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

故 $V \cap W$ 的基底爲

$$\left\{ \begin{bmatrix} 1 \\ 6 \\ 2 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 8 \\ -1 \\ -6 \\ -5 \end{bmatrix}, 2\begin{bmatrix} 1 \\ 6 \\ 2 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ -1 \\ -5 \\ -6 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 3 \\ 14 \\ 1 \\ -8 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 15 \\ 3 \\ -9 \\ 0 \end{bmatrix} \right\}$$

6. (10%) True or False. You must justify your answer mathematically.

(1) (5%) Two matrices that represent the same linear transformation $T : V \to V$ with respect to different bases are not necessarily similar.

(2) (5%) The standard basis for $\mathbb{R}^n$ will always make the coordinate matrix for the linear transformation T the simplest matrix possible.

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《解》 ☞

(1) False；設 α、β 爲 V 的有序基底，則

$$[T]_\beta = [I]_\alpha^\beta [T]_\alpha [I]_\beta^\alpha = ([I]_\beta^\alpha)^{-1} [T]_\alpha [I]_\beta^\alpha$$

(2) False；$\mathbb{R}^2$ 的標準基底爲 α，再令

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$c = -\det(A_2) = -\frac{1}{2}\begin{vmatrix} 0 & -2 \\ 2 & 4 \end{vmatrix} = -2$$

故選 (A)、(C)

3. Continued from Problem 2. Determine which of the following vectors are the eigenvectors of the matrix A_2 :

(A) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (B) $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ (C) $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ (D) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (E) $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$.

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《**解**》☞ 因

$$\det(A_2 - \lambda I) = (-1)^3(\lambda^3 - 4\lambda^2 + 5\lambda - 2) = 0$$

可解得 $\lambda = 1$、1、2，將 $\lambda = 1$ 代回 $(A_2 - \lambda I)X = 0$ 中可得

$$\begin{bmatrix} -2 & 2 & 2 \\ -1 & 1 & 1 \\ -2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

可得對應的特徵向量爲

$$X = c_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad (\ c_1、c_2\ 不全爲\ 0\ )$$

將 $\lambda = 2$ 代回 $(A_2 - \lambda I)X = 0$ 中可得

$$\begin{bmatrix} -3 & 2 & 2 \\ -1 & 0 & 1 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

可得對應的特徵向量爲

$$X = c_3\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad (c_3 \neq 0)$$

選 (A)、(C)、(D)、(E)

故

$$a_n''(t) + 2\beta a_n'(t) + n^2\, a_n(t) = 0$$

令 $a_n(t) = e^{mt}$ 代回 上式可得

$$m^2 + 2\beta m + n^2 = 0$$

故 $m = -\beta \pm \sqrt{n^2 - \beta^2}\, i \;\; (0 < \beta < 1)$，因此

$$a_n(t) = e^{-\beta t}(A_n \cos \sqrt{n^2 - \beta^2}\; t + B_n \sin \sqrt{n^2 - \beta^2}\; t)$$

則 PDE 的全解爲

$$u(x\,,\,t) = \sum_{n=1}^{\infty} e^{-\beta t}(A_n \cos \sqrt{n^2 - \beta^2}\; t + B_n \sin \sqrt{n^2 - \beta^2}\; t)\sin nx \qquad (1)$$

再由

$$u(x\,,\,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin nx$$

故

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

且

$$u_t(x\,,\,0) = 0 = \sum_{n=1}^{\infty}(-\beta A_n + \sqrt{n^2 - \beta^2}\; B_n)\sin nx$$

故 $B_n = \dfrac{\beta A_n}{\sqrt{n^2 - \beta^2}}$，將 A_n、B_n 代回 (1) 式即爲解。

4. (12%) The differential equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

can be used to describe a damped simple harmonic motion. Its solution can be written as the form of $x(t) = x_m e^{-\alpha t}\cos(\omega t + \phi)$, where x_m is the amplitude of the damped oscillator. Please solve this differential equation and find the α and ω in terms of m, b, k. **《喻超凡, 喻超弘** 102成大光電》

《解》 ☞ 令 $x = e^{\lambda t}$ 代回 ODE 中可得

$$m\lambda^2 + b\lambda + k = 0$$

故

$$\lambda = \frac{-b \pm \sqrt{4mk - b^2}\ i}{2m} = -\frac{b}{2m} \pm \sqrt{\frac{k}{m} - (\frac{b}{2m})^2}\ i = -\alpha \pm \omega\, i$$

其中 $\alpha = \dfrac{b}{2m}$ 、$\omega = \sqrt{\dfrac{k}{m} - (\dfrac{b}{2m})^2}$, 故

$$x(t) = e^{-\alpha t}(c_1 \cos \omega t + c_2 \sin \omega t) = x_m e^{-\alpha t} \cos(\omega t + \phi)$$

其中 $x_m = \sqrt{c_1^2 + c_2^2}$ 、$\phi = \tan^{-1}(\dfrac{-c_2}{c_1})$ 。

5. (1) (5%) $\vec{F} = [\sin^2 x\,,\ -y \sin 2x\,,\ 5z]$, S the surface of the box $|x| \le a$, $|y| \le b$, $|z| \le c$. Please evaluate the integral $\iint_S \vec{F} \cdot \vec{n}\, dA$.
   (2) (5%) $\vec{F} = [y\,,\ z^2\,,\ x^3]$, C the intersection of $x^2 + y^2 = 1$ and $z = y + 1$. Please evaluate the integral $\oint_C \vec{F} \cdot \vec{r}' ds$. (The line integral is clockwise as seen by a person standing at the origin.)
   (3) (5%) Please find a parametric representation of the following curve: Circle $\frac{1}{2}x^2 + y^2 = 1$, $z = y$

《喻超凡, 喻超弘 102成大光電》

《解》☞

(1) 由散度定理可知

$$\begin{aligned}\iint_S \vec{F} \cdot \vec{n}\, dA &= \iiint_V (\nabla \cdot \vec{F})\, dxdydz \\ &= \iiint_V (2 \sin x \cos x - \sin 2x + 5)\, dxdydz \\ &= \iiint_V 5\, dxdydz = 5(2a)(2b)(2c) = 40abc\end{aligned}$$

(2) 令 $x^2 + y^2 = 1$ 、$z = y + 1$ 相交的曲線 C , 所圍的區域為 S , 再令

$$\phi = z - y - 1 = 0$$

則 S 的單位法向量為

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{-\vec{j} + \vec{k}}{\sqrt{2}}$$

將 (1) 式兩端取共軛轉置，可得

$$(AX)^* = (\lambda X)^* \;\Rightarrow\; X^* A^* = \overline{\lambda}\, X^* \tag{2}$$

因 A 為 Skew–Hermitian 矩陣, 故滿足 $A^* = -A$，將其代入 (2) 式中，可得

$$-X^* A = \overline{\lambda}\, X^*$$

兩端後乘 X 可得

$$-X^* AX = \overline{\lambda}\, X^* X \;\Rightarrow\; -X^* \lambda X = \overline{\lambda}\, X^* X$$

故可得

$$(\lambda + \overline{\lambda}) X^* X = 0$$

因 $X^* X \neq 0$，則 $\lambda = -\overline{\lambda}$，即 λ 為純虛數或零。

7. (a) (5%) Please find all the singular points and the corresponding residues for $\dfrac{\sin z}{z^6}$.

(b) (5%) Please find the Cauchy principal value for $\displaystyle\int_{-\infty}^{\infty} \frac{x+5}{x^3 - x} dx$

(c) (5%) Please integrate $\dfrac{\cos z}{z^n}$ for $n = 1, 2, \cdots$ counterclockwise around $C : |z| = 1$.

《喻超凡, 喻超弘 102成大光電》

《解》 ☞

(a) 因 $z = 0$ 為 $f(z) = \dfrac{\sin z}{z^6}$ 的 5 階 pole，且

$$\text{Res} f(0) = \frac{1}{5!} \lim_{z\to 0} \frac{d^5}{dz^5} \{ z^6 \frac{\sin z}{z^6} \} = \frac{1}{5!} \lim_{z \to 0} (\cos z) = \boxed{\frac{1}{120}}$$

(b) 令 $f(z) = \dfrac{z+5}{z^3 - z}$，故 $F(z)$ 在實軸上具有 $z = 0$、± 1 的一階 poles，且

$$\text{Res} f(1) = \left. \frac{z+5}{3z^2 - 1} \right|_{z=1} = 3$$

$$\text{Res} f(0) = \left. \frac{z+5}{3z^2 - 1} \right|_{z=0} = -5$$

$$\text{Res} f(-1) = \left. \frac{z+5}{3z^2 - 1} \right|_{z=1} = 2$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=x^3} 2y\,dy\,dx$$
$$= \int_{x=0}^{x=1} y^2 \Big|_{y=0}^{y=x^3} dx$$
$$= \int_{x=0}^{x=1} x^6\,dx = \frac{1}{7}$$

因此

$$\int_{C_3} \vec{F}\cdot\vec{n}\,ds = \oint_{C_1+C_2+C_3} \vec{F}\cdot\vec{n}\,ds - \int_{C_1+C_2} \vec{F}\cdot\vec{n}\,ds = \frac{1}{7} - (-1) = \frac{8}{7}$$

8. (10%) Consider the following Lyapunov equation

$$XA + A^T X + Q = 0$$

where A is a (n-dimensional) real square matrix, and X, Q are real symmetric matrices.

(a) (5%) Suppose all eigenvalues of A have negative real parts. Show that

$$X = \int_0^\infty e^{A^T\tau} Q\, e^{A\tau}\,d\tau$$

is a solution to the Lyapunov equation.

(b) (5%) Suppose Q is positive definite and the Lyapunov equation has a positive definite solution X. Show that all eigenvalues of A have negative real parts.

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《解》☞

(a) 設 λ_1、λ_2、$\cdots$、λ_n 為方陣 A 的特徵值，且實部均為負值。若 A 可對解化，則存在一個非奇異的矩陣 S，使得

$$e^{At} = S \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} S^{-1}$$

令 $u = y^{-2}$, 故 $\frac{du}{dx} = (-2)y^{-3}\frac{dy}{dx}$, 代回上式可得

$$\frac{du}{dx} - 2u = -2x$$

積分因子為 $I = e^{-2x}$, 且

$$Iu = \int e^{-2x}(-2x)\,dx = xe^{-2x} + \frac{1}{2}e^{-2x} + c$$

故 ODE 的通解為

$$u = \frac{1}{y^2} = (x + \frac{1}{2}) + c\,e^{2x}$$

5. (20%) Assuming $z = re^{i\theta}$ and $\xi = \rho e^{i\phi}$, calculate following values.

(1) $\text{Re}\,\frac{z+\xi}{z-\xi}$ (10%)

(2) $\text{Im}\,\frac{z+\xi}{z-\xi}$ (10%)

《喻超凡, 喻超弘 102中山光電》

《解》 ☞ 因

$$\frac{z+\xi}{z-\xi} = \frac{(z+\xi)(\bar{z}-\bar{\xi})}{(z-\xi)(\bar{z}-\bar{\xi})} = \frac{z\bar{z}+\xi\bar{z}-z\bar{\xi}-\xi\bar{\xi}}{z\bar{z}-\xi\bar{z}-z\bar{\xi}+\xi\bar{\xi}} \tag{1}$$

因 $z\bar{z} = |z|^2 = r^2$, $\xi\bar{\xi} = |\xi|^2 = \rho^2$, 且

$$-\xi\bar{z} - z\bar{\xi} = -r\rho e^{i(\phi-\theta)} - r\rho e^{i(\theta-\phi)} = -r\rho(e^{i(\phi-\theta)} + e^{-i(\phi-\theta)}) = -2r\rho\,\cos(\phi-\theta)$$

$$\xi\bar{z} - z\bar{\xi} = r\rho e^{i(\phi-\theta)} - r\rho e^{i(\theta-\phi)} = r\rho(e^{i(\phi-\theta)} - e^{-i(\phi-\theta)}) = 2ir\rho\,\sin(\phi-\theta)$$

代回 (1) 式可得

$$\frac{z+\xi}{z-\xi} = \frac{z\bar{z}+\xi\bar{z}-z\bar{\xi}-\xi\bar{\xi}}{z\bar{z}-\xi\bar{z}-z\bar{\xi}+\xi\bar{\xi}} = \frac{r^2+2ir\rho\sin(\phi-\theta)-\rho^2}{r^2-2r\rho\cos(\phi-\theta)+\rho^2} \tag{2}$$

(1) 由 (2) 式可知

$$\text{Re}\,\frac{z+\xi}{z-\xi} = \frac{r^2-\rho^2}{r^2-2r\rho\cos(\phi-\theta)+\rho^2}$$

(2) 由 (2) 式可知

$$\text{Im}\,\frac{z+\xi}{z-\xi} = \frac{2r\rho\sin(\phi-\theta)}{r^2-2r\rho\cos(\phi-\theta)+\rho^2}$$

8. (8%) $z = x + iy$, 求z^3的實部與虛部。 《喻超凡, 喻超弘 102海洋光電》

《解》 ☞ $z^3 = (x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$, 故

$$\text{Re}(z^3) = x^3 - 3xy^2$$

$$\text{Im}(z^3) = 3x^2y - y^3$$

9. (13%) 請計算積分 $\int_0^\infty \frac{x^2+1}{x^6+1}dx$ 《喻超凡, 喻超弘 102海洋光電》

《解》 ☞ 令 $f(z) = \frac{z^2+1}{z^6+1}$, 由 $z^6 + 1 = 0$, 可得

$$z^6 = -1 = e^{(2k\pi+\pi)i} \Rightarrow z = e^{\frac{(2k\pi+\pi)i}{6}} \quad (k = 0\,,\,1\,,\,2\,,\,3\,,\,4\,,\,5)$$

爲 $f(z)$ 的奇點 , 只有 $z_1 = e^{i\frac{\pi}{6}}$ 、$z_3 = e^{i\frac{5\pi}{6}}$ 的一階 poles 在上半面 , $z_2 = e^{i\frac{\pi}{2}}$ 爲可棄奇點, 且其殘值分別爲

$$\text{Res}f(z_1) = \lim_{z\to z_1} \frac{z^2+1}{6z^5} = \frac{1}{6}(e^{2i\frac{\pi}{6}}+1)e^{-i\frac{5\pi}{6}} = \frac{1}{6}(-\frac{\sqrt{3}}{2} - i\frac{3}{2})$$

$$\text{Res}f(z_3) = \lim_{z\to z_3} \frac{z^2+1}{6z^5} = \frac{1}{6}(e^{2i\frac{5\pi}{6}}+1)e^{-i\frac{25\pi}{6}} = \frac{1}{6}(\frac{\sqrt{3}}{2} - i\frac{3}{2})$$

故

$$\begin{aligned}\int_0^\infty \frac{x^2+1}{x^6+1}dx &= \frac{1}{2}\int_{-\infty}^\infty \frac{x^2+1}{x^6+1}dx \\ &= \frac{1}{2}\cdot 2\pi i\{\text{Res}f(z_1) + \text{Res}f(z_3)\} \\ &= \frac{\pi}{2}\end{aligned}$$

3. (15%) Solve $\cos x\,(e^{2y} - y)\dfrac{dy}{dx} = e^{y}\sin 2x,\ y(0) = 0.$

《喻超凡, 喻超弘 102暨南應光》

《解》☞ 原式可改寫成

$$(e^{y} - ye^{-y})dy = \frac{\sin 2x}{\cos x}\,dx$$

兩端積分可得

$$e^{y} + ye^{-y} + e^{-y} = -2\cos x + c$$

再由 $y(0) = 0$, 可得 $1 + 1 = -2 + c$, 故 $c = 4$, 則 ODE 的特解為

$$e^{y} + ye^{-y} + e^{-y} = -2\cos x + 4$$

4. (15%) Solve $y'' + y = 4x + 10\sin x$, $y(\pi) = 0,\ y'(\pi) = 2.$

《喻超凡, 喻超弘 102暨南應光》

《解》☞

(1) 齊次解：令 $y = e^{mx}$ 代回 ODE 中可得

$$m^2 + 1 = 0 \ \Rightarrow\ m = \pm i$$

故

$$y_h(x) = c_1\sin x + c_2\cos x$$

(2) 特解：

$$\begin{aligned} y_p(x) &= \frac{1}{D^2+1}(4x) + \frac{1}{D^2+1}(10\sin x) \\ &= (1 - D^2 + \cdots)(4x) + 10 \times \frac{x}{2}(-\cos x) \\ &= 4x - 5x\cos x \end{aligned}$$

(3) 通解：

$$y(x) = y_h(x) + y_p(x) = c_1\sin x + c_2\cos x + 4x - 5x\cos x$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} u(\tau\,,\,0)\,\frac{y}{\pi\{y^2+(x-\tau)^2\}}\,d\tau \\
&= \int_{0}^{\infty} 20\,\frac{y}{\pi\{y^2+(x-\tau)^2\}}\,d\tau \\
&= \frac{20}{\pi}(-\tan^{-1}\frac{x-\tau}{y})\Big|_0^{\infty} \\
&= \frac{20}{\pi}(\frac{\pi}{2}+\tan^{-1}\frac{x}{y}) \\
&= 10-(-\frac{20}{\pi})\tan^{-1}\frac{x}{y}
\end{aligned}
$$

3. (16%) If the particular solution that satisfies the following equation and initial conditions :

$$xy''+3y'+25xy=0\ ,\ \ 0<x<\infty\ ,\ \ y(0)=12\ ,\ \ y'(0)=0$$

can be written as $y(x)=Ax^BJ_1(Cx)+Dx^BY_1(Cx)$. $A=?$ $B=?$ $C=?$ $D=?$

《喻超凡, 喻超弘 101台大電機 C》

《解》 ☞ 原式可改寫成

$$x^2y''+3xy'+25x^2y=0$$

令 $1-2\alpha=3$, 故 $\alpha=-1$, 令 $y=x^{-1}z(x)$ 代回 ODE 中，可得

$$x^2z''+xz'+(25x^2-1)z=0$$

故

$$z(x)=c_1J_1(5x)+c_2Y_1(5x)$$

因此

$$y(x)=x^{-1}z(x)=c_1\frac{J_1(5x)}{x}+c_2\frac{Y_1(5x)}{x}$$

因

$$\lim_{x\to0}y(x)=\lim_{x\to0}\{c_1\frac{J_1(5x)}{x}+c_2\frac{Y_1(5x)}{x}\}=12$$

故 $c_2=0$, 且

$$
\begin{aligned}
c_1\lim_{x\to0}\frac{J_1(5x)}{x} &= c_1\lim_{x\to0}5\,J_1'(5x)=5\,c_1\,J_1'(0) \\
&= 5\,c_1\frac{1}{2}\{J_0(0)-J_2(0)\} \\
&= 5\,\frac{c_1}{2}=12
\end{aligned}
$$

(3) 相異的特徵值對應的特徵函數相對權函數 $w = \dfrac{1}{1+x^2}$ 為正交。

8. 試求解下列的邊界值問題 (18%)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}\ ,\ 0 < x < 1\ ,\ t > 0$$

邊界條件：$u(0\,,\,t) = \sin t\ ,\ u(1\,,\,t) = 0,\ t > 0$

初始條件：$u(x\,,\,0) = 0\ ,\ 0 < x < 1.$　　《喻超凡, 喻超弘 101 台大生機》

《解》☞ 令 $u(x\,,\,t) = \phi(x\,,\,t) - x\sin t + \sin t$, 代回 ODE 中可得

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \phi}{\partial t} - x\cos t + \cos t$$

且

$$u(0\,,\,t) = \phi(0\,,\,t) + \sin t = \sin t \ \Rightarrow\ \phi(0\,,\,t) = 0$$

$$u(1\,,\,t) = \phi(1\,,\,t) - \sin t + \sin t = 0 \ \Rightarrow\ \phi(1\,,\,t) = 0$$

$$u(x\,,\,0) = \phi(x\,,\,0) = 0$$

因 $\phi(0\,,\,t) = \phi(1\,,\,t) = 0$, 故特徵函數為

$$\left\{\sin n\pi x\right\}_{n=1}^{\infty}$$

故

$$\phi(x\,,\,t) = \sum_{n=1}^{\infty} a_n(t)\sin n\pi x$$

且

$$(-x+1) = \sum_{n=1}^{\infty} q_n \sin n\pi x$$

其中

$$q_n = 2\int_0^1 (-x+1)\sin n\pi x\,dx = \frac{2}{n\pi}$$

代回 PDE 中可得

$$\sum_{n=1}^{\infty} a_n(t)(-n^2\pi^2)\sin n\pi x = \sum_{n=1}^{\infty} a_n'(t)\sin n\pi x + \cos t\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin n\pi x$$

2. 101 台聯系統 C 卷

1. (9%) Let $a > b > 0$, and define $M = a\boldsymbol{u}\boldsymbol{u}^T + bI_4$, where $\boldsymbol{u} = [1\ \ 2\ \ 2\ \ 1]^T$ and I_4 is the identity matrix of dimension 4×4.
   (1) (5%) Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ be the eigenvalues of M. Compute $\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4$.
   (2) (4%) Find the maximal singular value of matrix M.

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《解》☞

(1) 因

$$M\boldsymbol{u} = (a\boldsymbol{u}\,\boldsymbol{u}^T + bI_4)\boldsymbol{u} = a\boldsymbol{u}\,\boldsymbol{u}^T\boldsymbol{u} + b\,\boldsymbol{u} = (10a+b)\boldsymbol{u}$$

故 $\boldsymbol{u}$ 所對應的特徵值爲 $\lambda_1 = (10a+b)$, 再令

$$\boldsymbol{v} \in \textbf{span}\{\boldsymbol{u}\}^{\perp}$$

故 $< \boldsymbol{v}\,,\,\boldsymbol{u} >= \boldsymbol{u}^T\boldsymbol{v} = 0$, 因此

$$M\boldsymbol{v} = (a\boldsymbol{u}\boldsymbol{u}^T + bI_4)\boldsymbol{v} = b\,\boldsymbol{v}$$

則 $\lambda_2 = \lambda_3 = \lambda_4 = b$, 故

$$\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = (10a+b) + b - b - b = 10a$$

(2) 因 $M^T = (a\boldsymbol{u}\boldsymbol{u}^T + bI_4)^T = a\boldsymbol{u}\boldsymbol{u}^T + bI_4$, 故 M 爲對稱矩陣, 則 M 的奇異值即爲 M 的特徵值, 故 M 對大的奇異值爲 $(10a+b)$ 。

《解》 ☞ 因題目的初始條件爲 $y(1)=3$、$y'(1)=0$，故 ODE 的解只能對 $x=1$ 展開，因 $x=1$ 爲 ODE 的常點，又因題目只要求前6項的解，因此採用直接求解法，令 ODE 的解爲

$$y(x)=\sum_{n=0}^{\infty}\frac{y^{(n)}(1)}{n!}(x-1)^n \tag{1}$$

$$y''(x)=2(x-1)y'(x)-8y(x) \;\Rightarrow\; y''(1)=-8y(1)=-24$$
$$y'''(x)=2(x-1)y''(x)+2y'(x)-8y'(x) \;\Rightarrow\; y'''(1)=-6y'(1)=0$$
$$y^{(4)}(x)=2(x-1)y'''(x)+2y''(x)-6y''(x) \;\Rightarrow\; y^{(4)}(1)=-4y''(1)=96$$
$$y^{(5)}(x)=2(x-1)y^{(4)}(x)+2y'''(x)-4y'''(x) \;\Rightarrow\; y^{(5)}(1)=-2y'''(1)=0$$
$$y^{(6)}(x)=2(x-1)y^{(5)}(x)+2y^{(4)}(x)-2y^{(4)}(x) \;\Rightarrow\; y^{(6)}(1)=0$$
$$y^{(7)}(x)=2(x-1)y^{(6)}(x)+2y^{(5)}(x) \;\Rightarrow\; y^{(7)}(1)=2y^{(5)}(1)=0$$
$$\vdots$$

將求出的 $y''(1)$、$y'''(1)$、$y^{(4)}(1)$、$y^{(5)}(1)$、$\cdots$，代回 (1) 中，可得

$$\begin{aligned} y(x) &= y(1)+y'(1)(x-1)+\frac{y''(1)}{2!}(x-1)^2+\frac{y'''(1)}{3!}(x-1)^3 \\ &\quad +\frac{y^{(4)}(1)}{4!}(x-1)^4+\frac{y^{(5)}(1)}{5!}(x-1)^5+\frac{y^{(6)}(1)}{6!}(x-1)^6+\cdots \\ &= 3-\frac{24}{2}(x-1)^2+\frac{96}{4!}(x-1)^4 \\ &= -5+8x+12x^2-16x^3+4x^4 \\ &= c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5 \end{aligned}$$

故 $c_0=-5$、$c_1=8$、$c_2=12$、$c_3=-16$、$c_4=4$、$c_5=0$，本題之所能解出，是剛好

$$y^{(5)}(1)=y^{(6)}(1)=y^{(7)}(1)=\cdots=0$$

否則無法求出正確解。

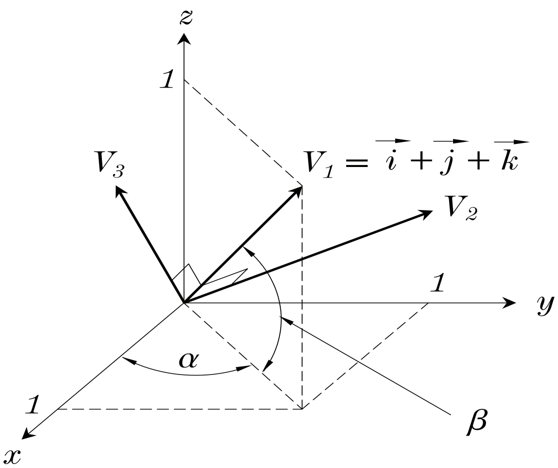
10. (10%) Consider the following boundary value problem for $u(x\,,\,t)$ with $0<x<\pi$ and $t>0$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2} \;,\; u(0\,,\,t)=u(\pi\,,\,t)=0\;,\; u(x\,,\,0)=f(x)\;,\; \left.\frac{\partial u}{\partial t}\right|_{t=0}=0 \tag{1}$$

(1) (5%) Assuming $f(x)=x$, determine its Fourier sine series on interval $[0\,,\,\pi)$.

(2) (5%) Solve the boundary value problem (1) for $u(x\,,\,t)$ when $f(x)=x$.

《喻超凡, 喻超弘 101台聯 C》



$\alpha = \frac{\pi}{4}$ 、$\beta = \tan^{-1}\frac{1}{\sqrt{2}}$

2. The matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix}$

   (a) (5%) Find the eigenvalues and the corresponding eigenvectors for the matrix A.

   (b) (5%) Calculate determinant of $(A^4 - 4A^3 - 3A^2 + 14A - 7I)$.

《**喻超凡, 喻超弘** 101交大光顯》

《解》 ☞

(a) 由 $\det(A - \lambda I) = 0$, 可得特徵值爲 $\lambda = 1$ 、4 、-2 , 將 $\lambda = 1$ 代回 $(A - \lambda I)X = 0$ 中可得

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

故對應的特徵向量爲

$$X = c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (c_1 \neq 0)$$

將 $\lambda = 4$ 代回 $(A - \lambda I)X = 0$ 中可得

$$\begin{bmatrix} -3 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

故對應的特徵向量爲

$$X = c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (c_2 \neq 0)$$

將 $\lambda = -2$ 代回 $(A - \lambda I)X = 0$ 中可得

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

故對應的特徵向量爲

$$X = c_3 \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix} \quad (c_3 \neq 0)$$

(b) 令 $g(A) = A^4 - 4A^3 - 3A^2 + 14A - 7I$，故 $g(A)$ 的特徵值爲

$$g(1) = 1 \text{ , } g(4) = 1 \text{ , } g(-2) = 1$$

故 $\det\{g(A)\} = g(1) \times g(4) \times g(2) = 1$ 。

3. The U be the subspace of $\mathbb{R}^4$ that consist of all vectors of the form

$$\begin{bmatrix} 2s - t \\ s \\ t \\ -5s + 2t \end{bmatrix}$$

(a) (5%) Find a basis for U.

(b) (5%) Find the orthogonal complement of U.

(c) (5%) Find the vector in U that is closest to $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

《喻超凡, 喻超弘 101交大光顯》

《提示》➥ 本題爲 96 年交大光顯的考題 。

《解》☞

將 $\lambda = 4$ 代回 $(A - \lambda I)X = 0$ 中可得

$$\begin{bmatrix} 0 & -3 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

故可得對應的特徵向量爲

$$X_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

令

$$S = [X_1 \ \ X_2] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

故

$$S^{-1}AS = D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

則

$$A^{1/2} = SD^{1/2}S^{-1} = S \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} S^{-1}$$

6. Let

$$B = \begin{bmatrix} -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \\ 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \end{bmatrix}$$

Mark the following statements True or False. Justify each answer.

(a) (5%) B is an invertible matrix.

(b) (5%) The linear transformation $x \mapsto Bx$ for any vector $x \in \mathbb{R}^4$ is one-to-one.

(c) (5%) Dimension of the null space of B is n.

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《解》☞

(a) T ; 因

$$\det(B) = \begin{vmatrix} -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \\ 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \end{vmatrix} = 2 \begin{vmatrix} -5 & -8 & 3 \\ 0 & 5 & -6 \\ 0 & 3 & -4 \end{vmatrix} = 20 \neq 0$$

(b) T ; 因 $\det(B) \neq 0$, 故 $\mathbf{N}(B) = \{\mathbf{0}\}$ 。

(c) F ; $\textbf{nullity}\,(B) = 0$